B. Review of Partial Fractions

When computing integrals and inverse Laplace transforms, *rational functions*, i.e. ratios of polynomials, arise:

$$R(s) = \frac{P(s)}{Q(s)} = \frac{p_n s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0}{q_m s^m + q_{m-1}s^{m-1} + \cdots + q_1 s + q_0}$$

It is useful to express $R(s)$ as a sum of simple fractions, this is called the *partial fractions expansion* of $R(s)$. Here's how to do that:

Step 0: If degree $P(s) \geq$ degree $Q(s)$, first perform a long division.

$$\textit{Example:} \quad \frac{s^5+1}{s^4+2s^2+1} = s - \frac{2s^3+s-1}{s^4+2s^2+1}\,.$$

Step 1: If the denominator hasn't already been factored, factor it completely.

$$\textit{Example:} \quad -\frac{2s^3+s-1}{s^4+2s^2+1} = -\frac{2s^3+s-1}{(s^2+1)^2}\,.$$

Step 2: For each factor in the denominator of the form $(s+a)^p$ include terms of the form

$$\frac{A_1}{(s+a)} + \frac{A_2}{(s+a)^2} + \cdots + \frac{A_p}{(s+a)^p}\,,$$

and for each term in the denominator of the form $(s^2+bs+c)^q$, include terms of the form

$$\frac{A_1 s + B_1}{(s^2+bs+c)} + \frac{A_2 s + B_2}{(s^2+bs+c)^2} + \cdots \frac{A_p s + B_p}{(s^2+bs+c)^q}$$

in the partial fractions expansion.

Examples:

$$-\frac{2s^3+s-1}{(s^2+1)^2} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+1)^2}\,,$$

$$\frac{2s^3-s^2+2s}{(s-1)^2(s^2+s+1)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{Cs+D}{(s^2+s+1)}\,,$$

$$\frac{3s^4+3s^3-3s^2-2s+4}{s^2(s-1)(s^2+2s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{Ds+E}{(s^2+2s+2)}\,.$$

$$\frac{3s^4+3s^3-3s^2-2s+4}{s^2(s-1)(s^2+2s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{Ds+E}{(s^2+2s+2)} + \frac{Fs+G}{(s^2+2s+2)^2}\,.$$

Step 3: Determine a system of equations for the unknown constants by collecting terms in the partial fractions expansion and equating the numerator of the result with the numerator of the original fraction.

$$\textit{Example:} \quad \frac{-2s^3-s+1}{(s^2+1)^2} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+1)^2} = \frac{As^3+Bs^2+(A+C)s+(B+D)}{(s^2+1)^2}\,.$$

So $A=-2 \quad B=0\,, \quad A+C=-1\,, \quad B+D=1$.

Step 4: Solve the system to determine the unknown constants.

Example (From Step 3): $A=-2\,, \quad B=0\,, \quad C=1\,, \quad D=1$.

Hence, $\displaystyle \frac{-2s^3-s+1}{(s^2+1)^2} = \frac{-2s}{(s^2+1)} + \frac{s+1}{(s^2+1)^2}.$