Math 307 - Section L
Win 2019
Final Exam
03/18/19

Name: Solutions

This exam contains 9 pages (including this cover page) and 9 problems. Put your first and last name on the top of this page.

You may *not* use your books, notes, or a **graphing** calculator on this exam.

You may use a **non-graphing** calculator on this exam.

You may use a single sheet of handwritten notes, with size not exceeding $11'' \times 8.5''$

Turn off all cellphones and electronic devices.

Do not open the exam until time.

Problem	Points	Score
1	8	
2	9	
3	8	
4	10	
5	6	
6	6	
7	6	
8	15	
9	12	
Total:	80	

1. (8 points) Find the explicit solution to the initial value problem

$$y' = \frac{t}{y\sqrt{t^2+1}}, \qquad y(0) = 2.$$

$$y\,dy = \frac{t}{\sqrt{t^2+1}}\,dt$$

$$\frac{1}{2}y^2 = \sqrt{t^2+1} + C$$

$$\frac{1}{2}\times 4 = 1 + C$$

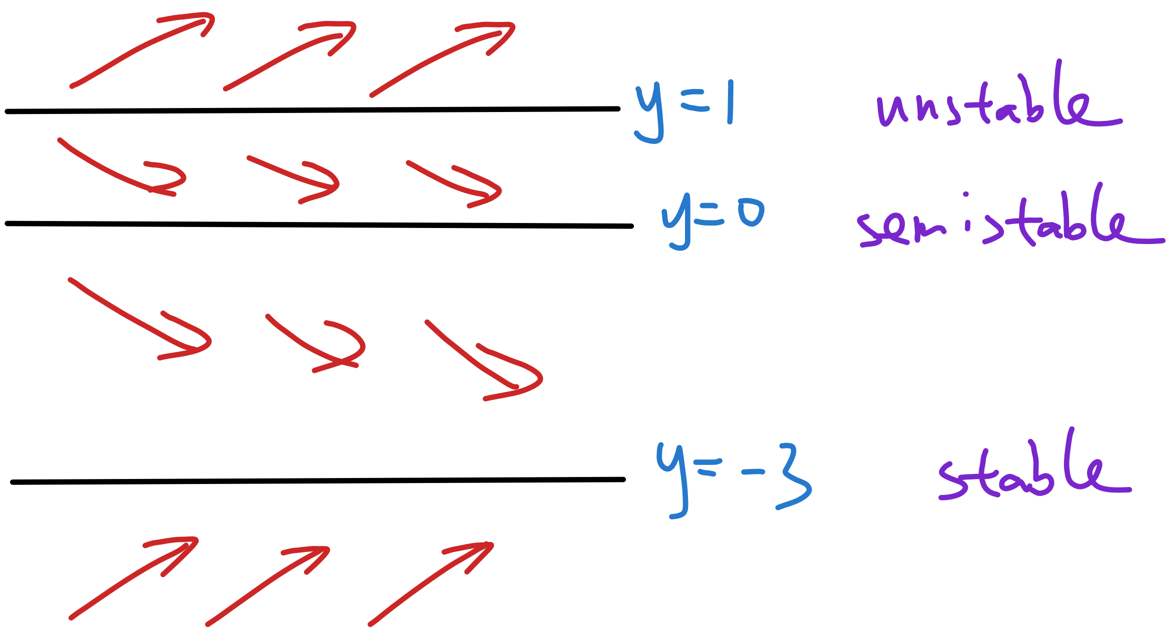
$$C = 1$$

$$y = \sqrt{2\sqrt{t^2+1} + 2}$$

2. (9 points) Find all equilibrium solutions to

$$y' = (y+3)y^2(y^2+1)(y-1)$$

and label them as stable, unstable or semistable. If $y(0) = \frac{1}{2}$, what is $\lim_{t\to+\infty} y(t)$? (It might be helpful to sketch a direction field)



If $y(0) = \frac{1}{2}$, $y(t) \to 0$

3. (8 points) Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings. There is a cup of ice water in a room with ambient temperature 72 Fahrenheit. The differential equation for the temperature of the ice water T can be written as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(72 - T).$$

Assume the initial temperature of the ice water is 30 Fahrenheit. After 10 minutes, the temperature becomes 51 Fahrenheit. Determine the constant k. (You need to specify the unit for k.)

$$\frac{dT}{dt} + kT = 72k$$

$$e^{kt} \cdot T' + e^{kt} \cdot kT = 72ke^{kt}$$

$$(e^{kt}T)' = 72ke^{kt}$$

$$e^{kt} \cdot T = 72e^{kt} + C$$

$$T = 72 + Ce^{-kt}$$

$C = -42$, $T = 72 - 42e^{-kt}$

$$T(10) = 72 - 42e^{-10t} = 51$$

$$42e^{-10k} = 21$$

$$e^{-10k} = \frac{1}{2}$$

$$-10k = \ln\frac{1}{2}$$

$$k = \frac{1}{10}\ln 2 \quad \text{min}^{-1}$$

4. (10 points) A 1 kg mass is attached to a spring with spring constant $10\,\text{kg/s}^2$ and is connected to a damper with damping coefficient $2\,\text{kg/s}$. No external force is applied. At time $t = 0$, the system is at position $y = -2\,\text{m}$ with initial velocity $y' = -4\,\text{m/s}$. Formulate an initial value problem for the position of the mass and solve it. Write the solution in the form of $Re^{\lambda t}\cos(\omega t - \varphi)$, $R > 0$.

$$y'' + 2y' + 10y = 0$$

$$y(0) = 2, \quad y'(0) = 4$$

$$y = C_1 e^{-t}\cos 3t + C_2 e^{-t}\sin 3t$$

$$y(0) = C_1 = -2$$

$$y'(0) = -C_1 + 3C_2 = -4 \Rightarrow \quad C_2 = -2$$

$$y = -2e^{-t}\cos 3t - 2e^{-t}\sin 3t$$

$$= 2\sqrt{2}\, e^{-t}\cos\left(3t - \frac{5\pi}{4}\right)$$

5. (6 points) Find a particular solution to the differential equation

$$y'' + 3y' + 2y = 2e^{-t}$$

$$r^2 + 3r + 2r = 0$$

$$r = -1, -2$$

$$Y(t) = Ate^{-t}$$

$$Y'(t) = Ae^{-t} - Ate^{-t}$$

$$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t}$$

$$Y''(t) + 3Y'(t) + 2Y(t)$$

$$= -2Ae^{-t} + Ate^{-t} + 3Ae^{-t} - 3Ate^{-t} + 2Ate^{-t}$$

$$= Ae^{-t} = 2e^{-t}$$

$$A = 2$$

$$Y(t) = 2te^{-t}$$

6. (6 points) Peter has one object, but he did not know the mass of the object. He found a (undamped) harmonic oscillator with mass 1 kg. The natural frequency of the harmonic oscillator is 1 rad/s. Then he replaced the mass with his own object, and remeasured the frequency. The frequency became 2 rad/s. Determine the mass of Peter's object.

$$w_A = \sqrt{\frac{k}{m_A}}$$

$$w_B = \sqrt{\frac{k}{m_B}}$$

$$\left(\frac{w_A}{w_B}\right)^2 = \frac{m_B}{m_A}$$

answer $\frac{1}{4}$ kg

$$\frac{\text{mass of Peter's object}}{1} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

7. (6 points) Find one pair of values for ω_0 and ω, such that $y(t) = A \sin t \sin 4t$ is a solution to the differential equation

$$y'' + \omega_0^2 y = \cos \omega t.$$

(OK to have negative w, w_0)

for some constant A.

method 1

$$\frac{w_0 + w}{2} = 4$$

$$\frac{w_0 - w}{2} = \pm 1$$

$$\Rightarrow \begin{cases} w = 5 \\ w_0 = 3 \end{cases} \text{ or } \begin{cases} w = 3 \\ w_0 = 5 \end{cases}$$

method 2 $y(t) = B(\cos 5t + \cos 3t)$ $\quad B = \frac{1}{2}A$

$y''(t) = -B(25 \cos 5t + 9 \cos 3t)$ $\quad w_0^2 = 25$

$-25B \cos 5t - 9B \cos 3t + Bw_0^2 \cos 5t \Rightarrow$ or 9

$+ w_0^2 B \cos 3t = \cos wt$

8. (15 points) The following parts are NOT related.

(a) (5 points) Find the inverse Laplace transform of

$$\frac{3s-9}{s^2+4s+13}$$

$$\frac{3(s+2)-15}{(s+2)^2+9} = 3\cdot\frac{s+2}{(s+2)^2+9} - 5\frac{3}{(s+2)^2+9}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$3e^{-2t}\cos 3t - 5e^{-2t}\sin 3t$$

(b) (5 points) Find the inverse Laplace transform of

$$\frac{e^{-2s}}{(s-1)^2(s-3)}$$

$$\frac{1}{(s-1)^2(s-4)} = -\frac{1}{4}\frac{1}{s-1} - \frac{1}{2}\frac{1}{(s-1)^2} + \frac{1}{4}\frac{1}{s-3} \overset{\mathcal{L}^{-1}}{\rightsquigarrow} -\frac{1}{4}e^{t} - \frac{1}{2}te^{t} + \frac{1}{4}e^{3t}$$

answer $$u_2(t)\left[-\frac{1}{4}e^{t-2} - \frac{1}{2}(t-2)e^{t-2} + \frac{1}{4}e^{3(t-2)}\right]$$

(c) (5 points) Calculate the convolution of $\sin t$ and $\sin 2t$.

Method 1. $\mathcal{L}\{(\sin t)*(\sin 2t)\} = \mathcal{L}\{\sin t\}\mathcal{L}\{\sin 2t\}$

$$= \frac{1}{s^2+1}\cdot\frac{2}{s^2+4}$$

$$(\sin t)*(\sin 2t) = \mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)(s^2+4)}\right\} = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t$$

Method 2. $$\int_0^t \sin(t-\tau)\sin 2\tau\, d\tau = \frac{1}{2}\int \cos(t-3\tau) - \cos(t+\tau)\, d\tau$$

$$= -\frac{1}{6}\sin(t-3\tau)\Big|_{\tau=0}^{\tau=t} - \frac{1}{2}\sin(t+\tau)\Big|_{\tau=0}^{\tau=t}$$

$$= \frac{1}{6}\sin 2t + \frac{1}{6}\sin t - \frac{1}{2}\sin 2t + \frac{1}{2}\sin t$$

9. (12 points) Solve the initial value problem

$$y'' + y = \begin{cases} 1, & 0 \le t < 3 \\ 5, & 3 \le t < 5 \\ 0, & t \ge 5 \end{cases}, \qquad y(0) = 0,\ y'(0) = 2.$$

$$y'' + y = 1 + 4u_3(t) - 5u_5(t)$$

$$s^2 Y(s) - 2 + Y(s) = 1 + 4\frac{e^{-3s}}{s} - 5\frac{e^{-5s}}{s}$$

$$(s^2+1)Y(s) = \frac{1}{s} + e^{-3s}\frac{4}{s} - 5\frac{e^{-5s}}{s} + 2$$

$$Y(s) = \frac{1}{s(s^2+1)} + e^{-3s}\frac{4}{s(s^2+1)} - e^{-5s}\frac{5}{s(s^2+1)} + \frac{2}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos t$$

$$y(t) = 1 - \cos t + 4u_3(t)[1 - \cos(t-3)]$$
$$- 5u_5(t)[1 - \cos(t-5)] + 2\sin t$$