Math 307 - Section L **Name:** Solutions
Win 2019
Exam 1
02/01/19

This exam contains 6 pages (including this cover page) and 5 problems. Put your first and last name on the top of this page.

You may *not* use your books, notes, or a **graphing** calculator on this exam.

You may use a **non-graphing** calculator on this exam.

Turn off all cellphones and electronic devices.

Do not open the exam until time.

Problem	Points	Score
1	9	
2	10	
3	13	
4	8	
5	10	
Total:	50	

1. (9 points) The following parts are NOT related.

(a) (3 points) Verify that $y(t) = \cos t$ is a solution to the differential equation $y'' + y = 0$.

$y' = -\sin t$

$y'' = -\cos t$

$y'' + y = -\cos t + \cos t = 0$

(b) (3 points) Find all equilibrium solutions for $y' = y \cos y$.

$y = 0$ or $\cos y = 0$

equilibrium solns: 0, $k\pi + \frac{1}{2}\pi$,

k any integer

(c) (3 points) Is the solution to the initial value problem unique: $ty' = y$, $y(0) = 0$? Justify your answer.

No. $y(t) = Ct$ are solutions for any constant C.

2. (10 points) Find the explicit solution to the initial value problem

$$y' = \frac{(y^2-4)\cos 3t}{y}, \quad y(0) = 3.$$

Here y is a function of t.

$$\frac{y}{y^2-4}\,dy = \cos 3t$$

$$\frac{1}{2}\ln|y^2-4| = \frac{1}{3}\sin 3t + C$$

$$\ln|y^2-4| = \frac{2}{3}\sin 3t + C$$

$$y^2-4 = C\cdot e^{\frac{2}{3}\sin 3t}$$

$$\text{I.C} \Rightarrow \quad C = 5$$

$$y^2 = 5e^{\frac{2}{3}\sin 3t} + 4$$

$$y = \pm\sqrt{5e^{\frac{2}{3}\sin 3t} + 4}$$

$$\text{I.C} \Rightarrow \quad y = \sqrt{5e^{\frac{2}{3}\sin 3t} + 4}$$

3. (13 points) Consider the initial value problem

$$y' = e^{-3t} + 2te^{-2t} - y, \quad y(0) = 1.$$

Here y is a function of t.

(a) (1 point) Is this equation linear or nonlinear? (You do not need to explain)

linear

(b) (2 points) What is $y'(0)$ and $y''(0)$?

$$y'(0) = e^{-3\cdot 0} + 2\cdot 0 e^{-2\cdot 0} - y(0) = 0$$

$$y''(t) = -3e^{-3t} + 2e^{-2t} - 4te^{-2t} - y'(t)$$

$$y''(0) = -3 + 2 - 0 = -1$$

(c) (8 points) Find the explicit solution.

$$y' + y = e^{-3t} + 2te^{-2t}$$

integrating factor e^t

$$(e^t y)' = e^{-2t} + 2te^{-t}$$

$$e^t y = -\frac{1}{2}e^{-2t} + \int 2te^{-t}dt$$

$$= -\frac{1}{2}e^{-2t} - \int 2t\, de^{-t}$$

$$= -\frac{1}{2}e^{-2t} - 2te^{-t} + 2\int e^{-t}dt$$

$$= -\frac{1}{2}e^{-2t} - 2te^{-t} - 2e^{-t} + C$$

I.C $\Rightarrow$ $1 = -\frac{1}{2} - 2 + C$, $C = \frac{7}{2}$

$$y = -\frac{1}{2}e^{-3t} - 2te^{-2t} - 2e^{-2t} + \frac{7}{2}e^{-t}$$

(d) (2 points) For the solution obtained in part (c), what is

$$\lim_{t\to+\infty} y(t) = 0$$

4. (8 points) Consider a free falling object with mass m, and assume the air resistance is proportional to v^3 (with constant of proportionality $\beta > 0$), where v is the velocity of the object. The constant of gravity is $g > 0$ (Do not use a number for g). Use the convention that v is positive if the object is going down.

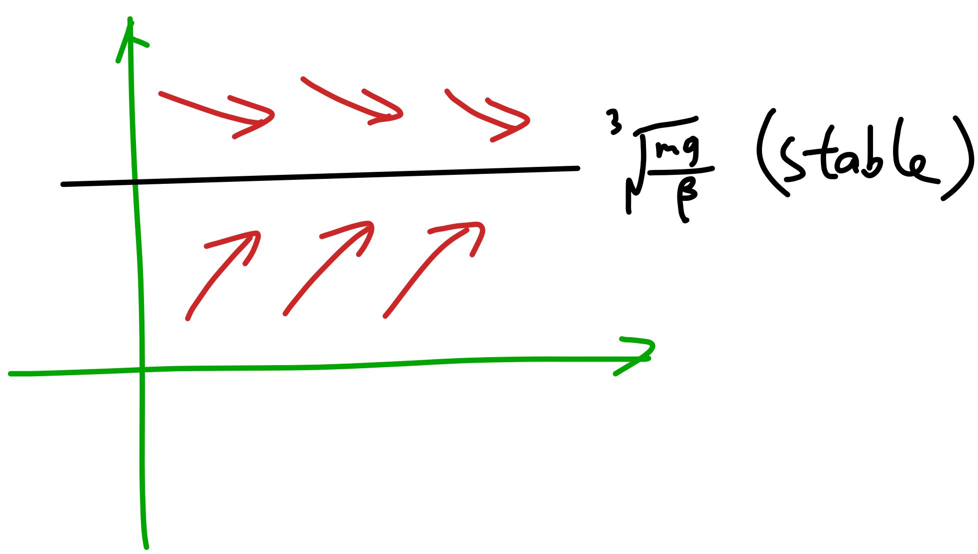
   (a) (4 points) Write down the differential equation for the velocity of the object as a function of time. You do not need to solve it.

$$m\frac{dv}{dt} = mg - \beta v^3$$

   (b) (4 points) Sketch a direction field and show the equilibrium solution, label it as stable or unstable. You need to find the value of the equilibrium in m, β, g.

$$\frac{dv}{dt} = g - \frac{\beta}{m}v^3$$

Equilibrium soln: $g - \frac{\beta}{m}v^3 = 0$. $v = \sqrt[3]{\frac{mg}{\beta}}$



5. (10 points) Suppose a tank initially (at $t = 0$) contains 100 liters (L) of **fresh** water. Suppose further that:

- Water flows into the tank at a rate of 2 liters per minute with a concentration of 10 gram (g) of salt per liter of water.
- Water flows out of the tank at a rate of 3 liters per minute.
- The salt-water mixture is well (perfectly) mixed.

(a) (5 points) Formulate an initial value problem for, $Q(t)$, the amount of salt in grams in the tank at time t (measured in minutes) before the tank is empty. Specify the domain of t.

$$\frac{dQ}{dt} = 20 - \frac{3Q}{100-t} \qquad t \in [0, 100)$$

$$Q(0) = 0$$

(b) (5 points) Solve the initial value problem for $Q(t)$.

$$\frac{dQ}{dt} + \frac{3Q}{100-t} = 20$$

Integrating factor $\mu(t) = e^{\int \frac{3}{100-t} dt} = e^{-3\ln|t-100|}$

$= e^{-3\ln(100-t)}$

$= (100-t)^{-3}$

notice the domain of t. $100-t>0$

$$[(100-t)^{-3}Q]' = 20(100-t)^{-3}$$

$$(100-t)^{-3}Q = 10(100-t)^{-2} + C$$

I.C $\Rightarrow$. $10 \times 100^{-2} + C$, $C = -10^{-3}$

$$Q = 10(100-t) - 10^{-3}(100-t)^3$$