1. (8 points) Find the explicit solution to the initial value problem

$$y^2\sqrt{t^2+1}y' - ty = 0, \qquad y(0) = -2.$$

Here y is a function of t.

$$\frac{dy}{dt} = \frac{ty}{y^2\sqrt{t^2+1}}$$

$$y\,dy = \frac{t}{\sqrt{t^2+1}}\,dt$$

$$\frac{1}{2}y^2 = (t^2+1)^{\frac{1}{2}} + C$$

$y(0) = -2 \Rightarrow$

$$2 = 1 + C$$

$$C = 1$$

$$y^2 = 2(t^2+1)^{\frac{1}{2}} + 2$$

$$y = \pm\sqrt{2(t^2+1)^{\frac{1}{2}} + 2}$$

$y(0) = -2 \Rightarrow$

$$y = -\sqrt{2(t^2+1)^{\frac{1}{2}} + 2}$$

2. (10 points) Find the solution to the initial value problem

$$t^3 y' = -4t^2 y + \sin t, \quad y(\pi) = 0, \quad t > 0.$$

Here y is a function of t.

$t^3 y' + 4t^2 y = \sin t$ multiply by t

$t^4 y + 4t^3 y = t \sin t$

$(t^4 y)' = t \sin t$

$t^4 y = \int t \sin t \, dt$

$= -\int t \, d\cos t$

$= -t\cos t + \int \cos t \, dt$

$= -t\cos t + \sin t + C$

IC $\Rightarrow$ $0 = \pi + C \Rightarrow C = -\pi$

$y = -\frac{\cos t}{t^3} + \frac{\sin t}{t^4} - \frac{\pi}{t^4}$

3. (10 points) Suppose a quantity $B(t) > 0$ is governed by the first order differential equation

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{1}{2}B\cos B - KB$$

where K is a constant such that $B = \frac{\pi}{3}$ is an equilibrium solution.

(a) (4 points) Find K and state whether $B = \frac{\pi}{3}$ is stable or unstable.

(b) (6 points) If $B(t)$ is the unique solution to the above differential equation satisfying $B(0) = 4\pi$, what is $\lim_{t\to+\infty} B(t)$? Justify your answer.

(a). $\frac{1}{2}B\cos B - KB = B\left(\frac{1}{2}\cos B - K\right)$

$$\frac{1}{2}\cos\frac{\pi}{3} - K = 0$$

$$\frac{1}{4} - K = 0$$

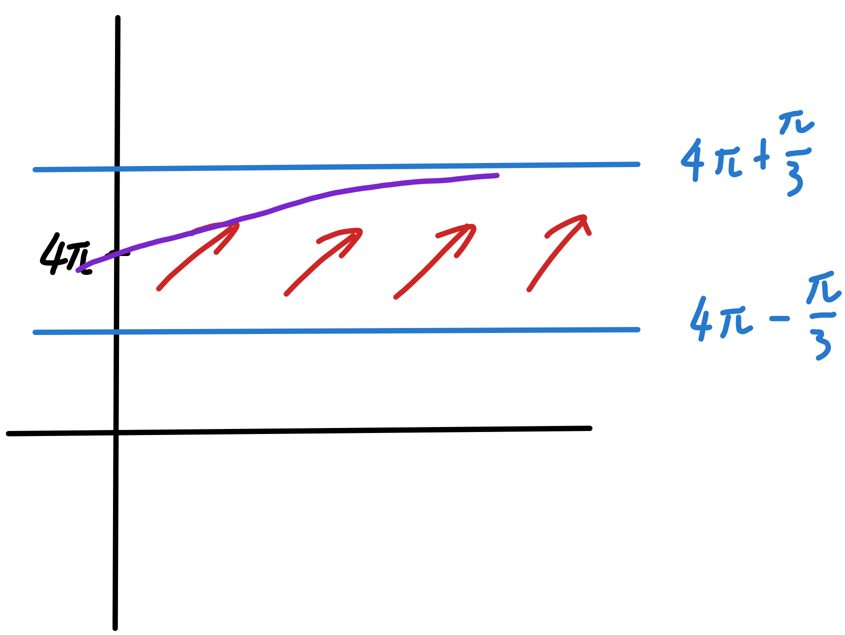
$$K = \frac{1}{4}$$

In a neighborhood of $\frac{\pi}{3}$, $B\left(\frac{1}{2}\cos B - \frac{1}{4}\right) > 0$, $B < \frac{\pi}{3}$

$B\left(\frac{1}{2}\cos B - \frac{1}{4}\right) < 0$ $B > \frac{\pi}{3}$

stable!

(b) Other equilibrium solutions: $2k\pi \pm \frac{\pi}{3}$, $k = 1, 2, 3, \cdots$



$$\lim_{t\to\infty} B(t) = \frac{13\pi}{3}$$

4. (10 points) The velocity $v(t)$ of a falling object with air resistance proportional to its velocity satisfies the differential equation

$$m\frac{dv}{dt} = -kv + mg$$

where g is the magnitude of gravitational acceleration, k is a constant which depends on the object (called the coefficient of air resistance) and m is the mass of the object.

Note: The convention we are using here is that v is negative when the body is ascending (going up) and positive when it is descending (going down).

(a) (7 points) Assume that $v(0) = 0$. Find a formula for $v(t)$.

Note: Your answer will be a formula in t, m, k and g. Do not use a numerical value for g, Simplify your answer.

(b) (3 points) An object with $m = 20$ kg, $k = 2$ kg/s started to fall with zero initial velocity and reached velocity 50 m/s at time t_0. Find the value of t_0. Assume $g = 10$ m/s^2.

(a) $$mv' + kv = mg$$

$$v' + \frac{k}{m}v = g$$

$$v'e^{\frac{k}{m}t} + \frac{k}{m}e^{\frac{k}{m}t}v = e^{\frac{k}{m}t}g$$

$$(e^{\frac{k}{m}t}v)' = e^{\frac{k}{m}t}g$$

$$e^{\frac{k}{m}t}v = \frac{mg}{k}e^{\frac{k}{m}t} + C$$

I.C. $\Rightarrow \frac{mg}{k} + C = 0 \Rightarrow C = -\frac{mg}{k}$

$$e^{\frac{k}{m}t}v = \frac{mg}{k}(e^{\frac{k}{m}t} - 1)$$

$$v = \frac{mg}{k}(1 - e^{-\frac{k}{m}t})$$

(b) $$v(t) = 100(1 - e^{-\frac{1}{10}t})$$

$$v(t_0) = 100(1 - e^{-\frac{1}{10}t_0}) = 50$$

$$1 - e^{-\frac{1}{10}t_0} = \frac{1}{2}$$

$$e^{-\frac{1}{10}t_0} = \frac{1}{2}$$

$$-\frac{1}{10}t_0 = \log\frac{1}{2}$$

$$t_0 = 10\log 2 \text{ second}$$

5. (12 points) Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings

   (a) (4 points) There is a cup of ice water in a room with ambient temperature T_s. If T_s is measured in Fahrenheit and t is measured in minutes, $T_s(t) = 77 + e^{-t}\sin\frac{t}{10}$. The initial temperature of the ice water is 30 Fahrenheit. Assume the **absolute value** of the proportionality constant K is 1 (with unit min^{-1}). Let $T(t)$ be the temperature of the ice water at time t. Formulate an initial value problem for T.

   (b) (6 points) What is the temperature of the ice water at time t?

   (c) (2 points) Determine $\lim_{t\to+\infty} T(t)$ and justify your answer.

(a). $\frac{dT}{dt} = 1\cdot(T_s - T)$

$\frac{dT}{dt} = 77 + e^{-t}\sin\frac{t}{10} - T$ $T(0) = 30$

(b) $T' + T = 77 + e^{-t}\sin\frac{t}{10}$

$e^{t}T' + e^{t}T = 77e^{t} + \sin\frac{t}{10}$ integrating factor e^{t}

$(e^{t}T)' = 77e^{t} + \sin\frac{t}{10}$

$e^{t}T = 77e^{t} - 10\cos\frac{t}{10} + C$

I.C $\Rightarrow$ $30 = 77 - 10 + C$

$\Rightarrow C = -37$

$T = 77 - 10e^{-t}\cos\frac{t}{10} - 37e^{-t}$

(c) $\lim_{t\to\infty} T(t) = 77.$ since $e^{-t} \to 0$ as $t \to \infty$