1. (10 points) Solve the initial value problem

$$y'' + 6y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = -2.$$

Write the solution in the form of $Re^{-\lambda t}\cos(\omega t - \varphi)$.

characteristic eqn

$$r^2 + 6r + 13 = 0$$

$$(r+3)^2 + 2^2 = 0$$

$$r = -3 \pm 2i$$

$$y = C_1 e^{-3t}\cos 2t + C_2 e^{-3t}\sin 2t$$

$$y' = -3C_1 e^{-3t}\cos 2t - 2C_1 e^{-3t}\sin 2t$$
$$\quad -3C_2 e^{-3t}\sin 2t + 2C_2 e^{-3t}\cos 2t$$

$$\text{I.C.} \Rightarrow \left.\begin{array}{l} C_1 = 2 \\ -3C_1 + 2C_2 = -2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} C_1 = 2 \\ C_2 = 2 \end{array}\right.$$

$$y = 2e^{-3t}(\cos 2t + \sin 2t)$$

$$R = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\left.\begin{array}{l} \cos\varphi = \frac{\sqrt{2}}{2} \\ \sin\varphi = \frac{\sqrt{2}}{2} \end{array}\right. \Rightarrow \varphi = \frac{\pi}{4}$$

$$y = 2\sqrt{2}\, e^{-3t}\cos\left(2t - \frac{\pi}{4}\right)$$

2. (8 points) Consider the linear homogeneous equation

$$t^2y'' - 5ty' + 8y = 0$$

(a) (5 points) Find all values of p such that $y(t) = t^p$ is a solution to the above equation.

$y'(t) = p\, t^{p-1}$

$y''(t) = p(p-1)t^{p-2}$

substitute into the eqn.

$p(p-1)t^p - 5pt^p + 8t^p = 0$

$p(p-1) - 5p + 8 = 0$

$p^2 - 6p + 8 = 0$

$p = 2$ or 4

(b) (3 points) Find the general solutions to the differential equation.

$y = C_1 t^2 + C_2 t^4$

3. (10 points) Find the general solutions to

$$y'' - y = e^t + \cos t.$$

characteristi eqn $r^2 - 1 = 0$, $r = \pm 1$

general soln for homogeneous eqn

$$C_1 e^t + C_2 e^{-t}$$

① special soln to $y'' - y = e^t$

e^t is homogeneous solution

$$Y(t) = Ate^t$$

$$Y'(t) = Ae^t + Ate^t$$

$$Y''(t) = 2Ae^t + Ate^t$$

$$Y''(t) - Y(t) = 2Ae^t + \cancel{Ate^t} - \cancel{Ate^t} = e^t$$

$A = \frac{1}{2}$ $\quad Y = \frac{1}{2} t e^t$

② special solution to $y'' - y = \cos t$

$Y = A\cos t$, $Y'' = -A\cos t$

$$Y'' - Y = -A\cos t - A\cos t = \cos t$$

$A = -\frac{1}{2}$ $\quad Y = -\frac{1}{2}\cos t$

$$y = C_1 e^t + C_2 e^{-t} + \frac{1}{2} t e^t - \frac{1}{2}\cos t$$

4. (10 points) An undamped mass spring system is released from equilibrium with a velocity of 6 m/s. The mass is 3 kg and it oscillates with an amplitude of 2 meters. There is no forcing. Find the spring constant k.

$$3y'' + ky = 0$$

$$y(0) = 0, \quad y'(0) = 6$$

$$w_0 = \sqrt{\frac{k}{3}}$$

$$y = A\cos w_0 t + B\sin w_0 t$$

$$\text{I.C.} \Rightarrow \begin{cases} A = 0 \\ w_0 B = 6 \end{cases}$$

$$y = \underbrace{\frac{6}{w_0}}_{\text{amplitude}} \sin w_0 t$$

$$\frac{6}{w_0} = 2, \quad w_0 = 3$$

$$k = 3w_0^2 = 27$$

5. (12 points) A 1kg mass is attached to a spring with spring constant 9 Newtons/m and is forced by an external force of $16 \sin 5t$ Newtons. At time $t = 0$, the system is at equilibrium position $y = 0$ with initial velocity $y' = -2$m/s. Formulate an initial value problem and solve it. Write the solution as a product of two trigonometric functions.

$$y'' + 9y = 16 \sin 5t$$

soln to homogeneous eqn.

$$C_1 \cos 3t + C_2 \sin 3t$$

special soln to inhomogeneous eqn

$$Y = A \sin 5t. \quad Y'' = -25 A \sin 5t$$

$$Y'' + 9Y = -16 A \sin 5t = 16 \sin 5t$$

$$A = -1$$

$$y = C_1 \cos 3t + C_2 \sin 3t - \sin 5t$$

I.C. $\Rightarrow$ $C_1 = 0$, $3C_2 - 5 = -2$, $C_2 = 1$

$$y = \sin 3t - \sin 5t$$

$$= 2 \cos \frac{3t+5t}{2} \sin \frac{3t-5t}{2}$$

$$= -2 \cos 4t \sin t$$