1. Find the Laplace Transforms of the following functions:

   (a) $\frac{1}{2}t^3 + e^t \cos 5t$

   (b) $f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ e^{-2t} - 1, & t \geq 1 \end{cases}$

   (c) $f(t) = \begin{cases} \cos t, & 0 \leq t < \frac{\pi}{2} \\ 0, & t \geq \frac{\pi}{2} \end{cases}$

(a). $\frac{1}{2} \cdot \frac{3!}{s^4} + \frac{s-1}{(s-1)^2+25} = \frac{3}{s^4} + \frac{s-1}{(s-1)^2+25}$

(b). $f(t) = u_1(t)\left(e^{-2t} - 1\right)$

$\mathcal{L}\{f(t)\} = e^{-s}\left[\mathcal{L}\{e^{-2t-2}\} - \mathcal{L}\{1\}\right]$

$= e^{-s}\left[e^{-2} \cdot \frac{1}{s+2} - \frac{1}{s}\right]$

(c) $f(t) = \cos t\left[1 - u_{\frac{\pi}{2}}(t)\right]$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos t\} - e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos(t + \frac{\pi}{2})\}$

$= \mathcal{L}\{\cos t\} + e^{-\frac{\pi}{2}s} \cdot \mathcal{L}\{\sin t\}$

$= \frac{s}{s^2+1} + e^{-\frac{\pi}{2}s}\frac{1}{s^2+1}$

2. Find the Inverse Laplace Transforms of the following functions:
   (a) $\frac{1}{s^2-5s+6}$
   (b) $\frac{e^{-s}}{s^2+6s+10}$
   (c) $\frac{1}{(s^2+1)(s^2+4)}$
   (d) $\frac{3}{(s+1)^2(s+4)}$
   (e) $\frac{e^{-2s}}{(s-1)^3}$

(a) $\frac{1}{s^2-5s+6} = \frac{1}{s-3} - \frac{1}{s-2} \xrightarrow{\mathcal{L}^{-1}} e^{3t} - e^{2t}$

(b) $\frac{e^{-s}}{(s+3)^2+1} \xrightarrow{\mathcal{L}^{-1}} u_1(t)\, e^{-3(t-1)} \sin(t-1)$

(c) $\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$

$\xrightarrow{\mathcal{L}^{-1}} \frac{1}{3} \cdot \sin t - \frac{1}{6}\sin 2t$

(d) $\frac{3}{(s+1)^2(s+4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+4}$

$3 = A(s+1)(s+4) + B(s+4) + C(s+1)^2$

$A = -\frac{1}{3}.\quad B = 1.\quad C = \frac{1}{3}$

$-\frac{1}{3}\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{3}\frac{1}{s+4} \rightsquigarrow -\frac{1}{3}e^{-t} + te^{-t} + \frac{1}{3}e^{-4t}$

(e) $\frac{1}{2}t^2e^t$ delayed by 2 $= \frac{1}{2}u_2(t)(t-2)^2 e^{t-2}$

3. Solve the Initial Value Problem using Laplace Transform

$$y'' + 4y' + 13y = 0, \qquad y(0) = 1,\ y'(0) = 2$$

$$s^2Y(s) - s - 2 + 4sY(s) - 4 + 13Y(s) = 0$$

$$(s^2+4s+13)Y(s) = s+6$$

$$Y(s) = \frac{s+6}{(s+2)^2+9}$$

$$= \frac{s+2}{(s+2)^2+9} + \frac{4}{3}\cdot\frac{3}{(s+2)^2+9}$$

$$= e^{-2t}\cdot\cos 3t + \frac{4}{3}e^{-2t}\sin 3t$$

4. Solve the Initial Value Problem

$$y' - 2y = \begin{cases} 0, & 0 \le t < 2 \\ 4(t-2), & t > 2 \end{cases}, \qquad y(0) = 3.$$

$$y' - 2y = 4u_2(t)(t-2)$$

$$sY(s) - 3 - 2Y(s) = 4e^{-2s} \cdot \frac{1}{s^2}$$

$$Y(s) = 4e^{-2s} \cdot \frac{1}{s^2(s-2)} + \frac{3}{s-2}$$

$$\frac{1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$$

$$1 = As(s-2) + B(s-2) + Cs^2$$

$$A = -\frac{1}{4}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{4}$$

$$\frac{1}{s^2(s-2)} \xrightarrow{\mathcal{L}^{-1}} -\frac{1}{4} - \frac{1}{2} \cdot t + \frac{1}{4}e^{2t}$$

$$y(t) = u_2(t)\left[-1 - 2(t-2) + e^{2(t-2)}\right] + 3e^{2t}$$

$$= \begin{cases} 3e^{2t} & 0 \le t < 2 \\ 3 - 2t - e^{2(t-2)} + 3e^{2t} & t \ge 2 \end{cases}$$

5. Solve the Initial Value Problem

$$y'' + y = \delta(t-\pi) + \delta(t-2\pi), \qquad y(0)=0,\ y'(0)=0.$$

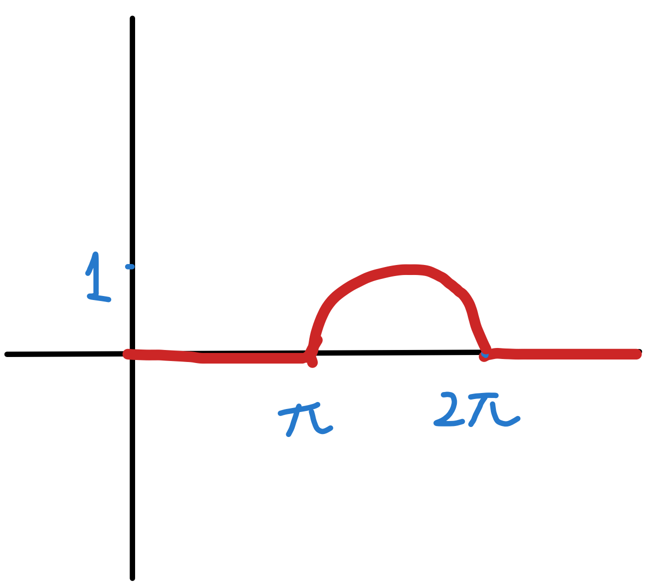
Here y is a function of t. Sketch a graph of the solution.

$$s^2 Y(s) + Y(s) = e^{-\pi s} + e^{-2\pi s}$$

$$Y(s) = e^{-\pi s} \cdot \frac{1}{s^2+1} + e^{-2\pi s} \cdot \frac{1}{s^2+1}$$

$$y(t) = u_\pi(t) \cdot \sin(t-\pi) + u_{2\pi}(t) \sin(t-2\pi)$$

$$= \begin{cases} 0 & 0 \le t < \pi \\ -\sin t & \pi \le t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$



6. Write the solution of the following IVP as a convolution integral

$$y'' + 4y = f(t), \qquad y(0) = 0,\ y'(0) = 0.$$

$$s^2 Y(s) + 4Y(s) = F(s)$$

$$Y(s) = \frac{1}{s^2+4} F(s)$$

$$y(t) = \frac{1}{2} \sin 2t * f(t)$$

$$= \frac{1}{2}\int_0^t \sin 2(t-\tau) f(\tau)\, d\tau$$