Differential Equations

- an equation for a function

  $y(t)$

  t: single variable

  Ordinary Differential Equation (ODE)

  $y(x)$. $y(x,t)$

  $x = (x_1, \cdots x_n)$. multiple variables

  Partial Differential Equation (PDE)

- containing derivatives of $y(t)$.

  $y(t) \quad y'(t) \quad y''(t). \cdots \quad y^{(n)}(t)$.

- modelling natural process.

  $y'(t)$: rate of change

  interests ~~rate~~ . velocity, population growth

  $y''(t)$: acceleration

Example: free falling object (no air resistance)

$F = mg$

$g = 9.8 m/s^2$

$y(t)$: position

$y'(t) = v(t)$ velocity

$v'(t) = y''(t) = a(t)$ acceleration.

Newton's law of motion

$$F = ma$$

$$\cancel{m}g = \cancel{m}v'(t)$$

$$v'(t) = g$$

solutions for this Diff Eq.

integrate in t

$$v(t) = gt + C$$

any constant

- infinitely many solutions.

- C can be determined if we know the initial velocity $v(0)$.

$$v(0) = g \cdot 0 + C$$

$$\Rightarrow C = v(0).$$

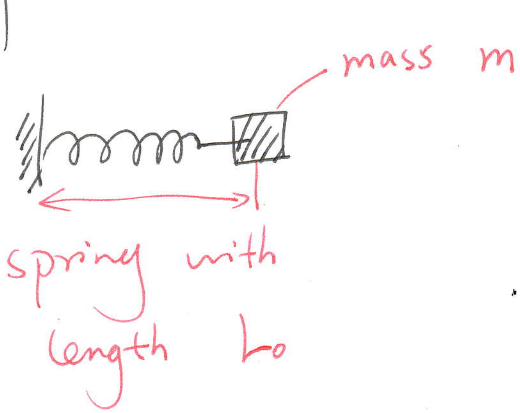
Problem. throw an object up with initial velocity $44.7\,m/s$. Find the velocity as a function of time

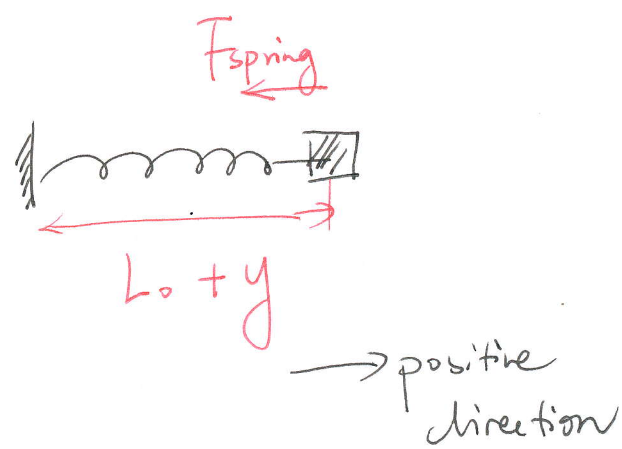
$$\begin{cases} v'(t) = 9.8 \\ v(0) = -44.7 \end{cases}$$

Initial Value Problem

$$v(t) = (\underbrace{9.8t}_{1\,sec} - 44.7)\,m/s$$

Example: Harmonic oscillator





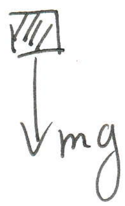
Hooke's law: $F_{spring} = -ky$

together with Newton's law

$$m\, y''(t) = -k\, y(t).$$

Example: Free ~~ob~~ falling object (with air resistance)

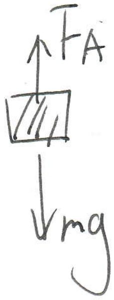
Assume: air resistance proportional to velocity



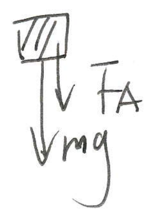
(opposite direction)

$$F_A = -\gamma\, v(t).$$

change with time



if $v \downarrow$



if $v \uparrow$

$$m\, v'(t) = mg - \gamma v(t).$$

$$v'(t) = g - \frac{\gamma}{m} v(t).$$

Let us assume $\gamma = 2\,kg/s$ $m = 10\,kg$

$$v'(t) = 9.8 - \frac{v(t)}{5}$$

Example: $p(t)$ population of mice

oversimplified assumptions:

- without predator: population growth proportional to the current population

$$\frac{dp}{dt} = rp. \qquad \text{mice/month}$$

$$r = 0.5/\text{month}.$$

- with owls, killing 450 mice/month

$$\frac{dp}{dt} = 0.5p - 450$$

linear ODEs

$$F(t, y(t), y'(t), \cdots, y^{(n)}(t)) = 0$$

$$a_n(t) y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \cdots + a_0(t) y(t) - g(t) = 0$$

An example of nonlinear ODE

Appendix Formulate the IVP in miles/hours

1 mile = 1610 meters

1 hour = 3600 seconds

$$v'(t) = 9.8 \ \frac{\text{meters}}{\text{sec}^2}$$

$$= 9.8 \times \frac{\text{miles}}{\text{hours}^2} \times \frac{\text{hours}^2}{\text{sec}^2} \times \frac{\text{meters}}{\text{miles}}$$

$$= 9.8 \times 3600^2 \times \frac{1}{1610} \ \frac{\text{miles}}{\text{hour}^2}$$

$$= 78887 \ \frac{\text{miles}}{\text{hour}^2}$$

$$V(0) = -44.7 \text{ m/s} = -100 \text{ mph}$$

$$\text{IVP} \quad \begin{cases} \dfrac{dv}{dt} = 78887 \ \dfrac{\text{miles}}{\text{hour}^2} \\ v(0) = 100 \text{ mph} \end{cases}$$