Differential Equations

- an equation for a function
  \[ y(t) \]
  \( t \): single variable
  Ordinary Differential Equation (ODE)

- \( y(x), y(x,t) \)
  \( x = (x_1, \ldots, x_n) \): multiple variables
  Partial Differential Equation (PDE)

- containing derivatives of \( y(t) \)
  \[ y(t), y'(t), y''(t), \ldots, y^{(n)}(t) \]

- modeling natural process
  \( y'(t) \): rate of change
  \( y''(t) \): acceleration
  \( y'''(t) \): population growth
  \( y''''(t) \): velocity
Example: free falling object (no air resistance)

\[ F = mg \]

\[ g = 9.8 \text{ m/s}^2 \]

\[ y(t): \text{position} \]
\[ y'(t) = v(t): \text{velocity} \]
\[ v'(t) = y''(t) = a(t): \text{acceleration} \]

Newton's law of motion

\[ F = ma \]

\[ mg = m v'(t) \]
\[ v'(t) = g \]

Solutions for this Diff Eq.

Integrate in \( t \)

\[ v(t) = gt + C \]

(any constant)

\[ \text{infinitely many solutions.} \]
\textbf{C} can be determined if we know the initial velocity \( v(0) \).

\[ v(t) = g \cdot 0 + C \]

\[ \Rightarrow C = v(0) \]

Problems: throw an object up with initial velocity \( 44.7 \text{ m/s} \). Find the velocity as a function of time

\[ v(t) = 9.8 \]

\[ v(0) = -44.7 \]

\[ \text{Initial Value Problem} \]

\[ v(t) = \left(9.8 \frac{t}{1 \text{sec}} - 44.7\right) \text{ m/s} \]

\textbf{Example: Harmonic Oscillator}

\begin{itemize}
  \item mass \( m \)
  \item spring with length \( L_0 \)
  \item \( F_{\text{spring}} \)
  \item \( L_0 + y \) → positive direction
\end{itemize}
Hooke's law: \( F_{\text{spring}} = -ky \)

together with Newton's law

\[ m y''(t) = -ky(t) \]

Example: Free falling object (with air resistance)

Assume: air resistance proportional to velocity

(\text{opposite direction})

\[ F_A = -\gamma v(t) \]

\( \text{change with time} \)

\[ \begin{align*}
\text{if } v & \downarrow \\
F_A &= \gamma v(t) \\
\text{if } v & \uparrow \\
F_A &= \gamma v(t)
\end{align*} \]

\[ m v'(t) = mg - \gamma v(t) \]

\[ v'(t) = g - \frac{\gamma}{m} v(t) \]

Let us assume \( v = 2 \text{ kg/s} \) \( m = 10 \text{ kg} \)

\[ v'(t) = 9.8 - \frac{v(t)}{5} \]
Example: \( p(t) \) population of mice

Oversimplified assumptions:

without predator: population growth proportional to the current population

\[
\frac{dp}{dt} = rp \quad \text{mice/month}
\]

\( r = 0.5 \text{ /month} \)

with owls, killing 450 mice/month

\[
\frac{dp}{dt} = -0.5p - 450
\]

Linear ODEs

\[
F(t, y, y', \ldots, y^{(n)}(t)) = 0
\]

\[
a_n y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \ldots + a_1(t) y'(t) + a_0(t) y(t) - g(t) = 0
\]

An example of non-linear ODE
Appendix Formulate the IVP in miles/hours

1 mile = 1610 meters

1 hour = 3600 seconds

\[ v'(t) = 9.8 \, \frac{\text{meters}}{\text{sec}^2} \]

\[ = 9.8 \times \frac{\text{miles}}{\text{hours}^2} \times \frac{\text{hours}^2}{\text{sec}^2} \times \frac{\text{meters}}{\text{miles}} \]

\[ = 9.8 \times 3600 \times \frac{1}{1610} \, \frac{\text{miles}}{\text{hour}^2} \]

\[ = 78887 \, \frac{\text{miles}}{\text{hour}^2} \]

\[ v(0) = -44.7 \, \text{m/s} = -100 \, \text{mph} \]

\[ \text{IVP} \]

\[ \frac{dv}{dt} = 78887 \, \frac{\text{miles}}{\text{hour}^2} \]

\[ v(0) = 100 \, \text{mph} \]