Lecture 12. (Feb. 6)

Second Order Differential Equations

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$$

Linear 2nd D.E.,

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

homogeneous Eq. if $G(t) = 0$

inhomogeneous Eq if $G(t) \neq 0$

Focus on the case where P, Q, R are constants

Homogeneous 2nd order D.E. with constant coefficients

$$ay'' + by' + c = 0$$

Example. $y'' - y = 0$

observe $y_1 = e^t$ is a solution

$y_2 = e^{-t}$ is another solution

$2 \cdot e^t$. $5 \cdot e^{-t}$ are solutions.

$y = C_1 e^{t} + C_2 e^{-t}$ is solution, for any constants C_1, C_2

Two unknowns, need two initial conditions

(IVP) $\begin{cases} y'' - y = 0 \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$

Suppose the solution is of the form

$$y = C_1 e^{t} + C_2 e^{-t}$$

Use I.C.

$$C_1 + C_2 = 2$$

$$y' = C_1 e^{t} - C_2 e^{-t}$$

$$C_1 - C_2 = -1$$

$$C_1 = \frac{1}{2}, \quad C_2 = \frac{3}{2}$$

$$y = \frac{1}{2} e^{t} + \frac{3}{2} e^{-t}$$

Next, let us consider

$$a y'' + b y' + c y = 0$$

If we found two independent solutions y_1, y_2, then

$y = C_1 y_1 + C_2 y_2$ is a solution.

Proof. $a(C_1 y_1 + C_2 y_2)'' + b(C_1 y_1 + C_2 y_2)' + c(C_1 y_1 + C_2 y_2)$

$$= C_1 (a y_1'' + b y_1' + c y_1) + C_2 (a y_2'' + b y_2'' + c y_2) = 0$$

Seek solutions of the form $y = e^{rt}$

$$\begin{aligned} & a y'' + b y' + c y \\ &= a r^2 e^{rt} + b r e^{rt} + c e^{rt} \\ &= (ar^2 + br + c) e^{rt} \end{aligned}$$

$y' = re^{rt}$. $y'' = r^2 e^{rt}$

e^{rt} never zero. $\Rightarrow \underline{ar^2 + br + c = 0}$

Characteristic equation

Roots of the characteristic equation, a, b, c real. $a \neq 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three situations,
① $b^2 - 4ac > 0$. two distint real roots
② $b^2 - 4ac = 0$ one real root
③ $b^2 - 4ac < 0$ two disitnct complex roots

For the D.E., (Treat 3 sitution separately)

①. two roots r_1, r_2, solution of the form

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$e^{r_1 t}$, $e^{r_2 t}$ two independent solns.

②. one root r, one solution found.

$$y_1 = e^{rt}$$

Need to find another one. To be discussed later.

③ Two distinct complex roots r_1, r_2. Tentatively,

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \qquad (C_1, C_2 \text{ complex})$$

want to find real-valued y.

Example: $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$

① Find general solutions.

characteristic eqn.

$$r^2 + 5r + 6 = 0$$

two distinct roots, $r_1 = -2$, $r_2 = -3$

General solutions

$$y = C_1 e^{-2t} + C_2 e^{-3t}$$

② Use I.C. to determine C_1, C_2

$$y(0) = C_1 + C_2 = 2$$

$$y'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t}$$

$$y'(0) = -2C_1 - 3C_2 = 3$$

$$C_1 = 9, \quad C_2 = -7$$

$$y(t) = 9e^{-2t} - 7e^{-3t}$$

Example. $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$

characteristic equation

$6r^2 - 5r + 1 = 0$, $r_1 = \frac{1}{2}$. $r_2 = \frac{1}{3}$

general solution

$$y = C_1 e^{\frac{t}{2}} + C_2 e^{\frac{t}{3}}$$

$$y' = \frac{C_1}{2} e^{\frac{t}{2}} + \frac{C_2}{3} e^{\frac{t}{3}}$$

I.C. $\Rightarrow$ $C_1 + C_2 = 4$

$$\frac{C_1}{2} + \frac{C_2}{3} = 0$$

$C_1 = -8$. $C_2 = 12$

$$y = -8 e^{\frac{t}{2}} + 12 e^{\frac{t}{3}}$$