Lecture 13. (Feb. 8)

$y'' + y = 0.$

Characteristic equation

$r^2 + 1 = 0$

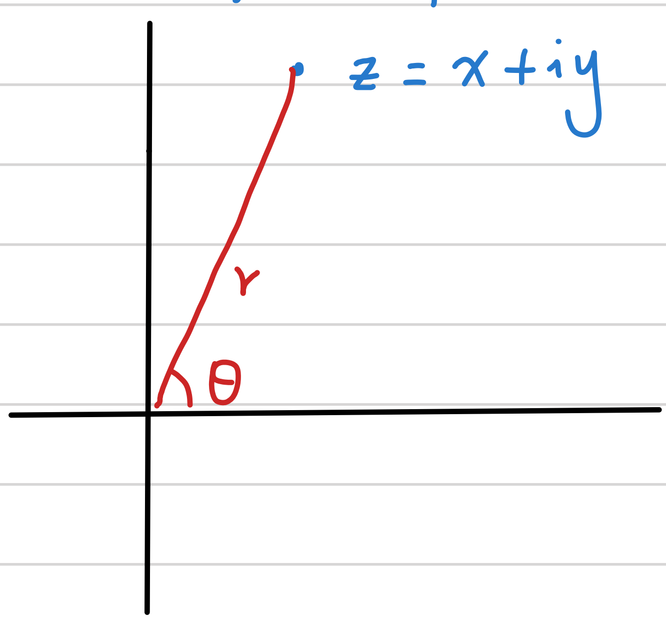
$r_1 = i. \quad r_2 = -i$

Two solutions

$e^{it} \quad e^{-it}$

How to make sense of e^{it}, e^{-it}

Review of complex numbers



x: real part of z

y: imaginary part

$x = r\cos\theta. \quad y = r\sin\theta$

$z = x + iy$

$= r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

with properly defined $e^{i\theta}$

$= re^{i\theta}$

Properties of e^{rt} for complex r

① $e^{r0} = 1$

② $\frac{d}{dt} e^{rt} = r \cdot e^{rt}$

(Theorem) If a function $g(t)$ satisfying, r real

$$g(0) = 1$$
$$\frac{dg}{dt} = rg \qquad (**)$$

Then $g(t) = e^{rt}$

We need to define e^{it} satisfying $(**)$ with $r = i$.

$$\frac{d}{dt}(\cos t + i \sin t) = -\sin t + i \cos t$$
$$i(\cos t + i \sin t) = i \cos t - \sin t$$

Euler's formula : $e^{it} = \cos t + i \sin t$

Rigorous definition

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \cdots \quad \text{(Taylor series about } t=0\text{)}$$

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots + \frac{(it)^n}{n!} + \cdots$$

$i^2 = -1 \quad i^4 = 1$

$i^3 = -i$

$$= 1 + it - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \cdots$$
$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)$$
$$= \cos t + i \sin t.$$

Usual properties for exp function hold

Go back to the equation

$$y'' + y = 1$$

two complex-valued solution

$$e^{it} = \cos t + i\sin t$$

$$e^{-it} = \cos t + i\sin(-t)$$

$$= \cos t - i\sin t$$

Notice, real and imaginary parts give us two (independ) real solutions $\cos t$, $\sin t$

More generally. consider $ay'' + by' + c = 0$

$b^2 - 4ac < 0$, two complex roots

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu$$

Two solutions:

$$e^{(\lambda+i\mu)t} \qquad e^{(\lambda-i\mu)t}$$

$$e^{(\lambda+i\mu)t} = e^{\lambda t} \cdot e^{i\mu t} = e^{\lambda t}(\cos\mu t + i\sin\mu t)$$

$$e^{(\lambda-i\mu)t} = e^{\lambda t} e^{-i\mu t} = e^{\lambda t}(\cos\mu t - i\sin\mu t)$$

Real part of the first soln: $e^{\lambda t}\cos\mu t = y_1$

Imaginary part $e^{\lambda t}\sin\mu t = y_2$

y_1, y_2 are two real solns

The second complex soln give the same things.

Theorem:

If $y = y_1 + i y_2$ is a solution, then

(y_1, y_2 real-valued)

y_1, y_2 are solutions

Proof: $ay'' + by' + cy = 0$

$$a(y_1 + iy_2)'' + b(y_1 + iy_2)' + c(y_1 + iy_2) = 0$$

$$ay_1'' + by_1' + cy_1 + i(ay_2'' + by_2' + cy_2) = 0$$

$$\Rightarrow \quad ay_1'' + by_1' + cy_1 = 0$$

$$ay_2'' + by_2' + cy_2 = 0.$$

Example: $y'' + 4y' + 20y = 0$, $y(0) = 2$, $y'(0) = 1$

characteristic equation

$$r^2 + 4r + 20 = 0$$

$$r = \frac{-4 \pm \sqrt{4^2 - 4\times 20}}{2}$$

$$= \frac{-4 \pm \sqrt{-64}}{2}$$

$$= \frac{-4 \pm 8i}{2}$$

$$= -2 \pm 4i$$

Complex solutions

$$e^{(-2+4i)t} = e^{-2t} e^{4it} = e^{-2t}(\cos 4t + i \sin 4t)$$

$$e^{(-2-4i)t} = e^{-2t} e^{-4it} = e^{-2t}(\cos 4t - i \sin 4t)$$

Two independent real solns

$$y_1 = e^{-2t} \cos 4t$$

$$y_2 = e^{-2t} \sin 4t$$

General solns: $y = C_1 e^{-2t} \cos 4t + C_2 e^{-2t} \sin 4t$

I.C. $\Rightarrow$. $C_1 = 2$

$$y' = -2C_1 e^{-2t}\cos 4t - 4C_1 e^{-2t}\sin 4t$$
$$-2C_2 e^{-2t}\sin 4t + 4C_2 e^{-2t}\cos 4t$$

$$y'(0) = -2C_1 + 4C_2 = 1$$

$$C_2 = \frac{5}{4}$$

$$y = 2e^{-2t}\cos 4t + \frac{5}{4}e^{-2t}\sin 4t$$

If the roots of characteristic polynomial are $\lambda \pm i\mu$. two independent solns

$$e^{\lambda t}\cos \mu t, \quad e^{\lambda t}\sin \mu t$$

$$y(t) = C_1 e^{\lambda t}\cos \mu t + C_2 e^{\lambda t}\sin \mu t$$