Lecture 14. (February 13)

$ay'' + by' + cy = 0$

Case III: $b^2 - 4ac = 0$

$\lambda = \frac{-b + \sqrt{b^2-4ac}}{2a} = -\frac{b}{2a}$

one real root λ.

$y_1 = e^{\lambda t}$, need to find y_2

Claim: $y_2 = te^{\lambda t}$ is a solution

y_1, y_2 are independent

Let us verify the claim

$y_2' = e^{\lambda t} + \lambda t e^{\lambda t}$

$y_2'' = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 t e^{\lambda t} = 2\lambda e^{\lambda t} + \lambda^2 t e^{\lambda t}$

$ay_2'' + by_2' + cy_2$

$= a(2\lambda e^{\lambda t} + \lambda^2 t e^{\lambda t}) + b(e^{\lambda t} + \lambda t e^{\lambda t}) + cte^{\lambda t}$

$= (a\lambda^2 + b\lambda + c)te^{\lambda t} + (2a\lambda + b)e^{\lambda t}$

$= 0$

General solutions $y = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$

Example: $y'' + 2y' + y = 0$, $y'(0) = 2$, $y'(0) = 1$

characteristic eqn: $r^2 + 2r + 1 = 0$

$$r = -1$$

General solns $y = C_1 e^{-t} + C_2 t e^{-t}$

I. C. $\Rightarrow$ $y(0) = C_1 = 2$

$$y' = -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t}$$

$$y'(0) = -C_1 + C_2 = 1$$

$$C_2 = 3$$

$$y = 2e^{-t} + 3te^{-t}$$

Summary of homogeneous second order differential equations with constant coefficients

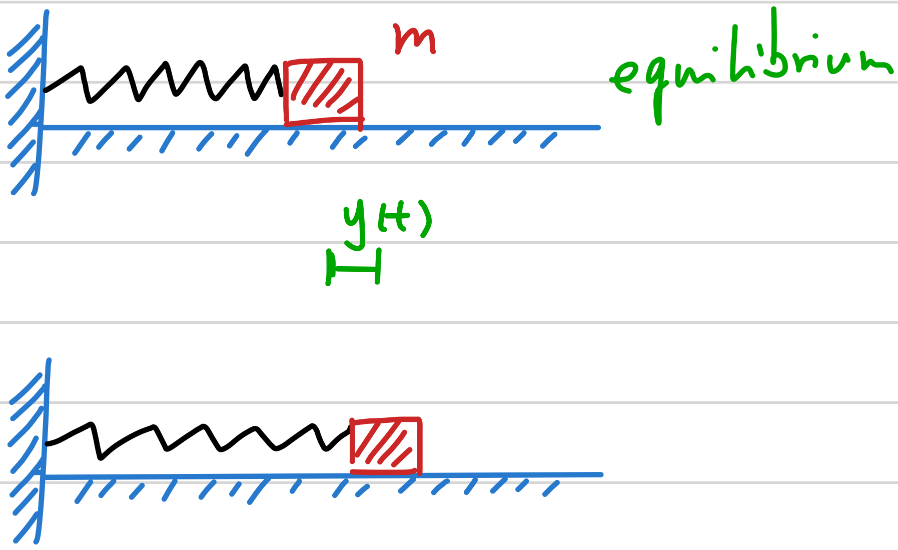
$$ay'' + by' + cy = 0$$

If $b^2 - 4ac > 0$, $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$b^2 - 4ac = 0$ $y = C_1 e^{rt} + C_2 t e^{rt}$

$b^2 - 4ac < 0$ $y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$

Harmonic oscillator.



$w_0 = \sqrt{\frac{k}{m}}$: frequency

$$m\,y''(t) = F_{spring} = -k\,y(t) \quad , k > 0$$

D.E. $m y''(t) + k y(t) = 0$

characteristic eqn.

$$m r^2 + k = 0$$

$$r = \pm\sqrt{\frac{k}{m}}\, i$$

$w_0 = \sqrt{\frac{k}{m}}$: natural frequency

I.C. $y(0) = y_0$ (initial position) $y'(0) = v_0$ (initial velocity)

Solutions of the form

$$y = A\cos w_0 t + B\sin w_0 t$$

A, B depend on y_0, v_0

want to write $A\cos w_0 t + B\sin w_0 t$ in the form

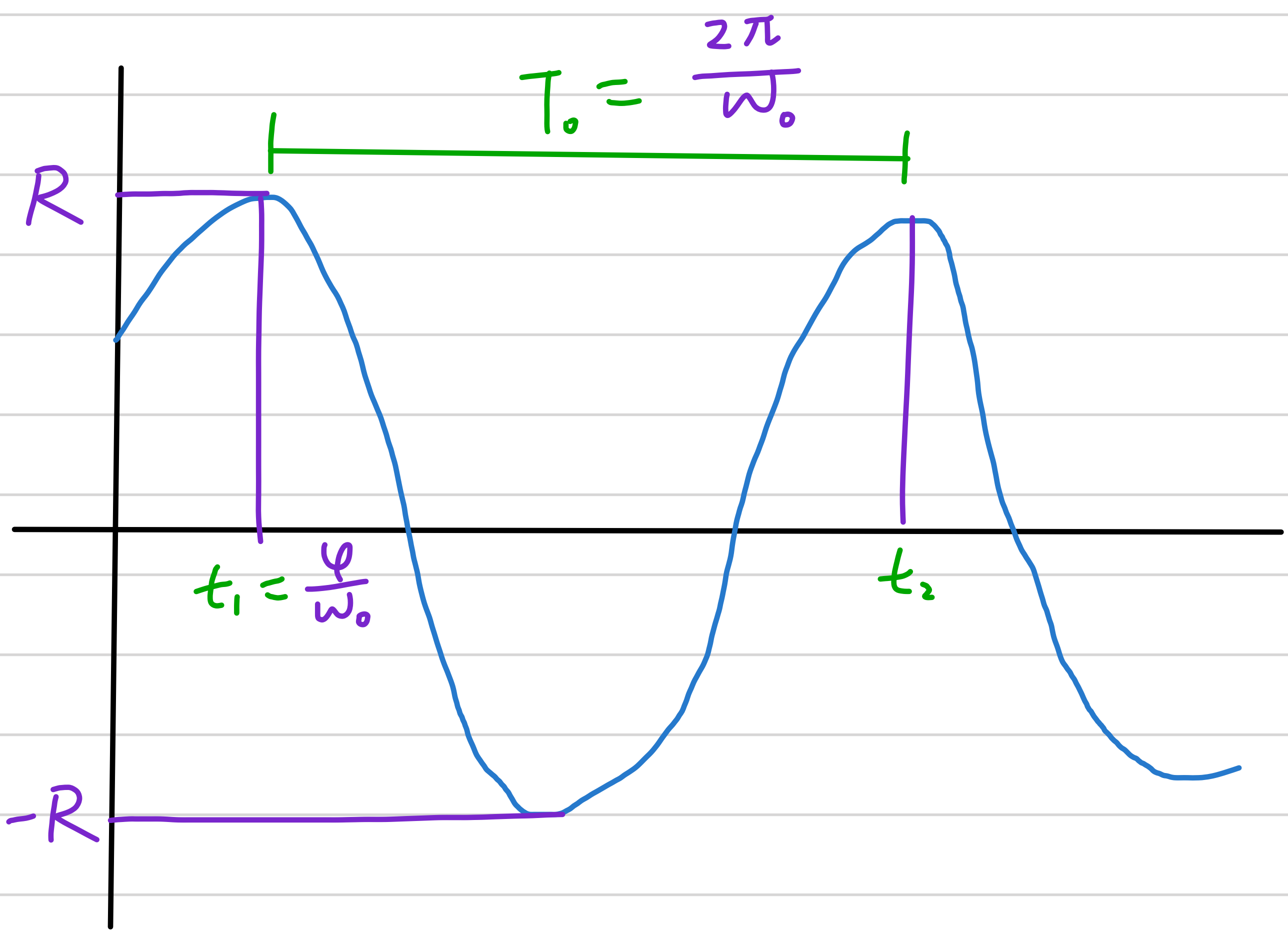
$$R\cos(w_0 t - \varphi)$$

R = amplitude $\quad R>0$

w_0 = frequency

φ = phase $\quad \varphi \in [0, 2\pi)$

Figure of $R\cos(w_0 t - \varphi)$



$w_0 t_1 - \varphi = 0$

$w_0 t_2 - \varphi = 2\pi$

$t_1 = \frac{\varphi}{w_0}$, $\quad w_0(t_2 - t_1) = 2\pi$

period : $T_0 = t_2 - t_1 = \frac{2\pi}{w_0}$ $\quad$ (period × frequency $= 2\pi$)

$$R\cos(\omega_0 t-\varphi) = R\cos\omega_0 t\cos\varphi + R\sin\omega_0 t\sin\varphi$$
$$= (R\cos\varphi)\cos\omega_0 t + (R\sin\varphi)\sin\omega_0 t$$

Recall $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Just need $R\cos\varphi = A$, $R\sin\varphi = B$

$$R^2(\cos^2\varphi + \sin^2\varphi) = A^2+B^2$$
$$R = \sqrt{A^2+B^2}$$

Example: $y = 3\cos 2t + 4\sin 2t$

$$R = \sqrt{3^2+4^2} = 5$$
$$5\cos\varphi = 3$$
$$5\sin\varphi = 4$$

$\cot\varphi = \dfrac{\cos\varphi}{\sin\varphi} = \dfrac{3}{4}$ (two angles in $[0, 2\pi)$ have $\cot = \frac{3}{4}$)

properties of arccot:
domain $(-\infty, +\infty)$
range $(0, \pi)$

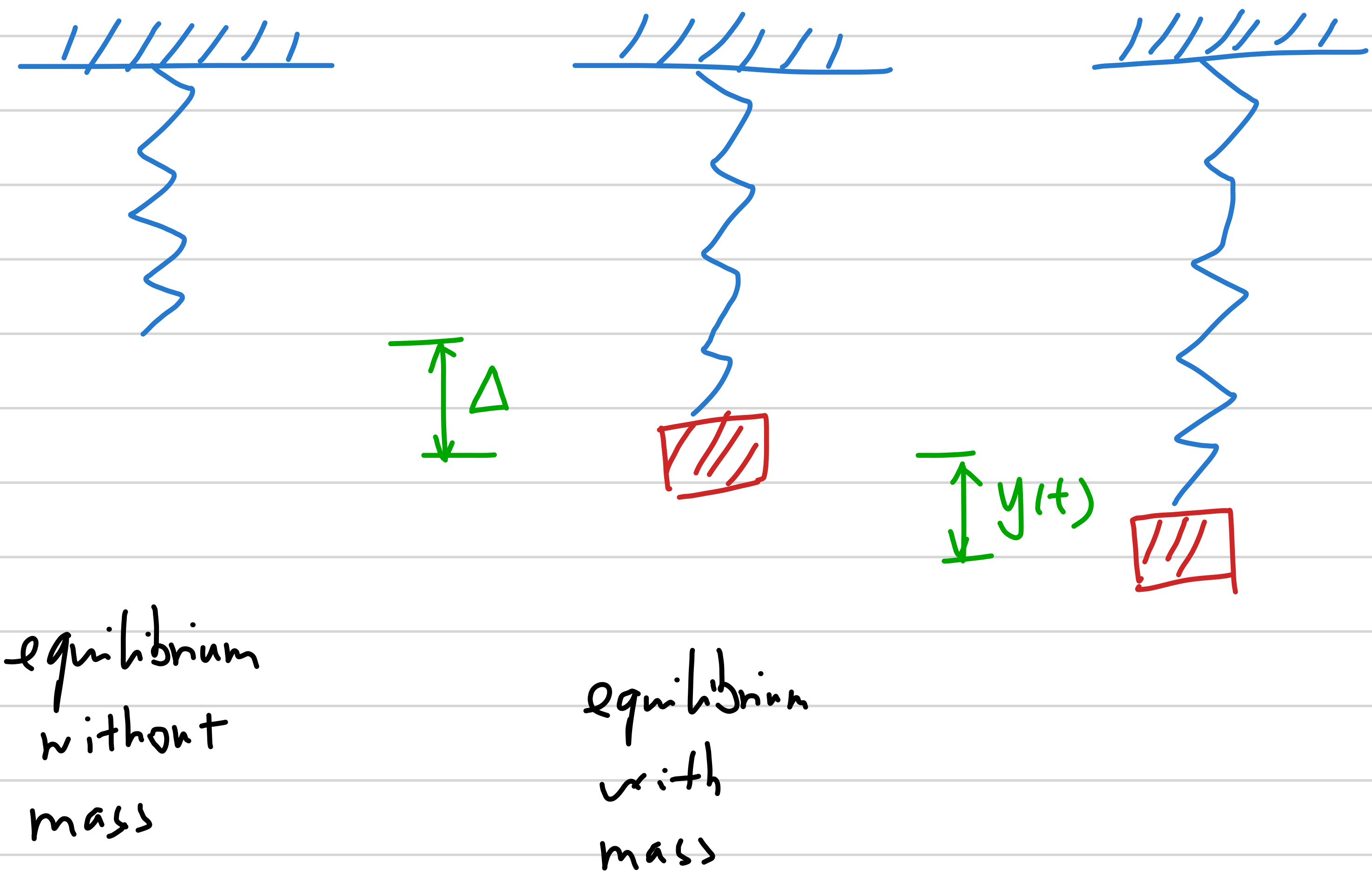
$\varphi = \operatorname{arccot}\frac{3}{4}$ or $\operatorname{arccot}\frac{3}{4} + \pi$

$\in [0, \frac{\pi}{2})$

take this since $\cos\varphi > 0$, $\sin\varphi > 0$

$$y = 5\cos(2t - \operatorname{arccot}\tfrac{3}{4})$$

Appendix



(*) $k\Delta = mg$ (equilibrium with mass)

$$my'' = F_{total} = F_{spring} + F_{gravity}$$
$$= -k(\Delta + y) + mg$$
$$= -k\cancel{\Delta} - ky + \cancel{mg} \quad \text{cancel by } (*)$$
$$= -ky$$

Same equation