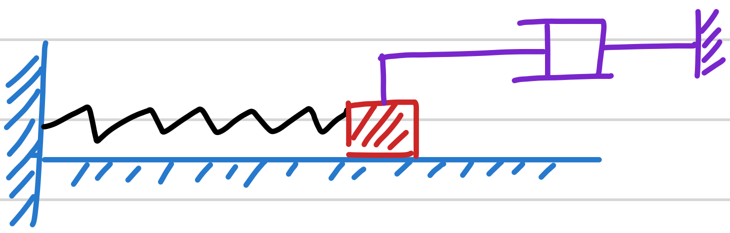
Lecture 15 (February 15)

Add a damper



Damper, provides a resistance proportional to the velocity

γ: damping coefficient

$F_{damping} = -\gamma y'$

Damped Harmonic Oscillator

$F_{total} = F_{damping} + F_{spring}$

$$my'' = -\gamma y' - ky$$

usually written as

$$my'' + \gamma y' + ky = 0$$

$m, \gamma, k > 0$

Roots $r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

If r real, $r < 0$

r complex, real part < 0

$b^2 - 4ac = \gamma^2 - 4mk$

Different behaviors when the signs of $\gamma^2 - 4mk$ are different.

$\gamma^2 - 4mk > 0$: $\gamma > 0$, two real roots $r_1, r_2 < 0$

over-damped

$\gamma^2 - 4mk = 0$ $\gamma > 0$, one real root $r < 0$

critically damped

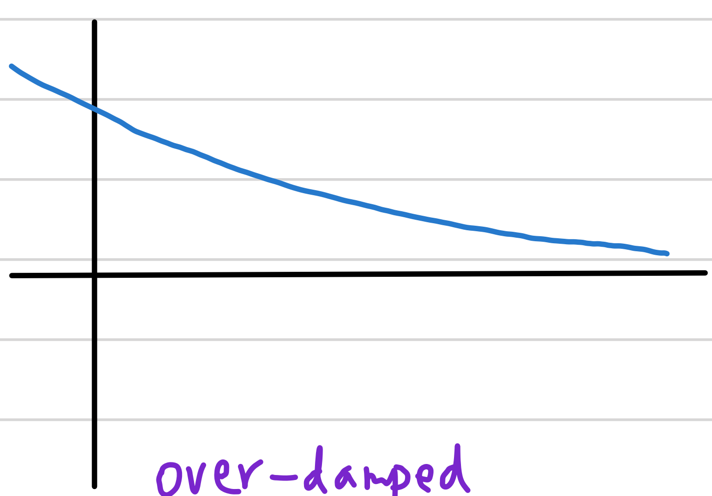
$\gamma^2 - 4mk < 0$ two complex roots

$\lambda \pm i\mu$, $\lambda < 0$

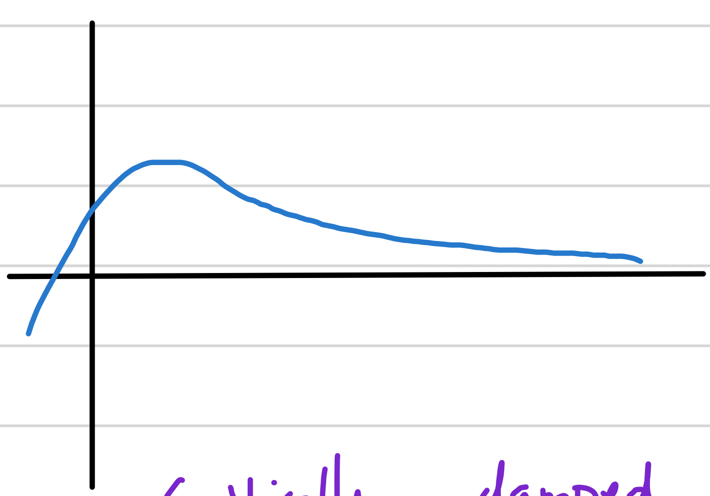
under damped

$\gamma = 0$. two complex roots $i\mu$

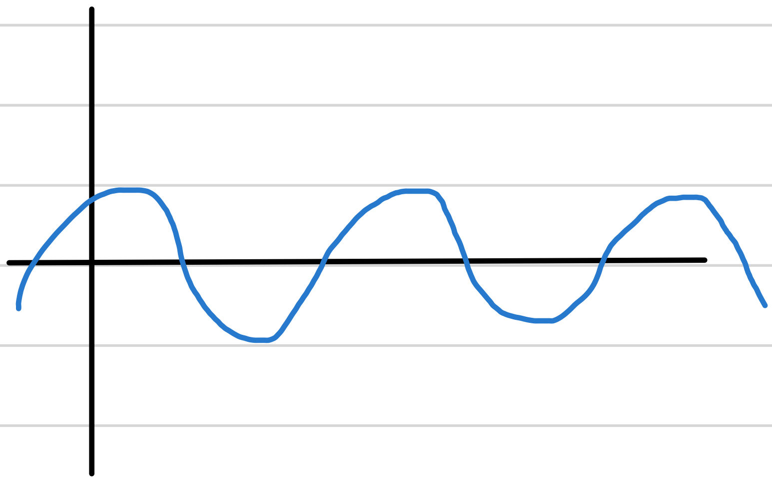
un-damped.



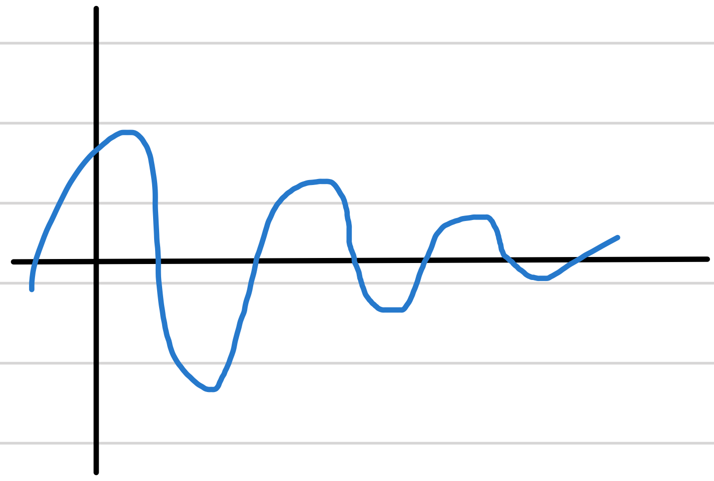
over-damped



Critically damped



undamped



Under-damped

More on the case $\gamma^2 - 4mk < 0$

$$\lambda = -\frac{\gamma}{2m}, \quad \mu = \frac{\sqrt{4mk-\gamma^2}}{2m} = w$$

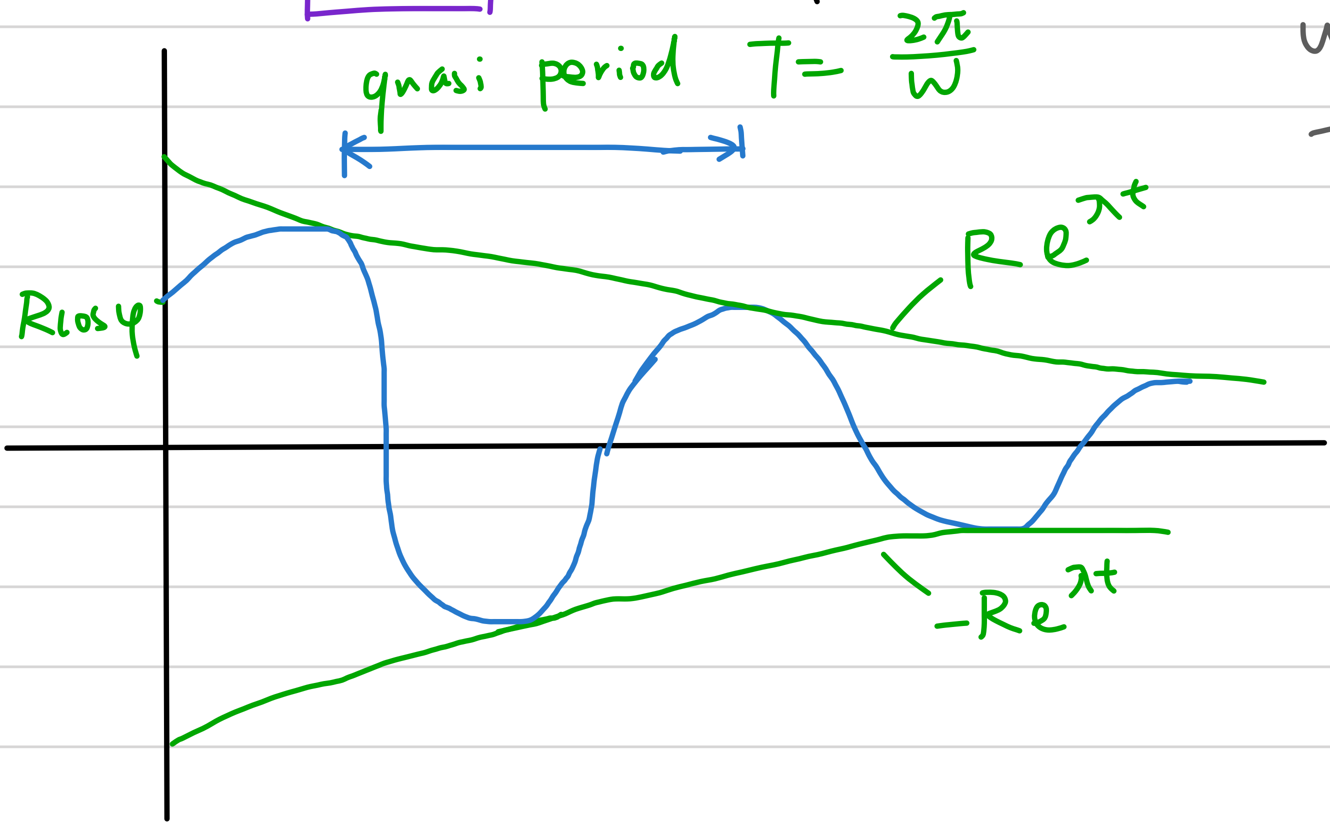
(quasi frequency)

($\neq w_0$ damping changes the frequency)

The solution can be written as

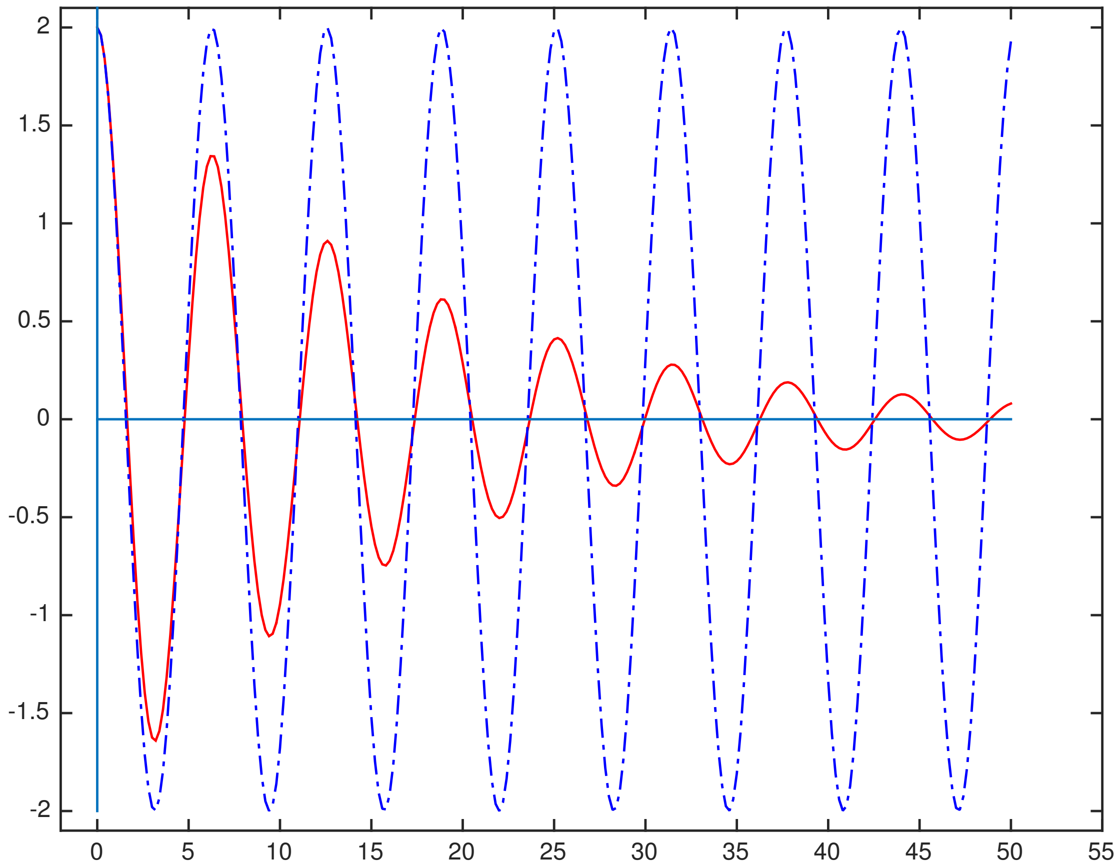
$$y = e^{\lambda t}(A\cos wt + B\sin wt)$$

$$= \underbrace{\boxed{Re^{\lambda t}}}_{\text{amplitude}}\cos(wt-\varphi)$$



$w < w_0$

$T > T_0$



Red solid line: solution to

$$y'' + y = 0,\ y(0) = 2,\ y'(0) = 0$$

Blue dashed line: solution to

$$y'' + 0.125y' + y = 0,\ y(0) = 2,\ y'(0) = 0$$

Small damping (γ small)

$$e^{\lambda t}\cos(\omega t - \varphi)$$

$$\omega = \frac{\sqrt{4mk - \gamma^2}}{2m} \sim \sqrt{\frac{k}{m}} = \omega_0$$

Inhomogeneous 2nd order D.E. with constant coefficients

$$ay'' + by' + cy = G(t)$$

$$L(y) = ay'' + by' + cy.$$

L linear.

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$

$$L(cy_1) = cL(y_1)$$

$L(y) = 0$: homogeneous eqn.

general soln $c_1 y_1 + c_2 y_2$

$L(y) = G(t)$; inhomogeneous eqn

one special solution $Y(t)$.

If y is a solution to the inhomogeneous eqn

i.e. $L(y) = G(t)$. then

$$y = c_1 y_1 + c_2 y_2 + Y$$

Proof. $L(y - Y) = L(y) - L(Y)$

$= G(t) - G(t)$

$= 0$

$y - Y$ is the solution to the homogeneous eqn.

Only need to find one special solution Y.

Method of undetermined coefficients

(need to guess the form of Y)

Example: $y'' - 3y' - 4y = 3e^{2t}$

Guess:

$$Y(t) = Ae^{2t}$$

then determine A

$$Y'(t) = 2Ae^{2t}$$

$$Y''(t) = 4Ae^{2t}$$

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$-6A = 3$$

$$A = -\frac{1}{2}$$

$$Y = -\frac{1}{2}e^{2t}$$