Lecture 16 (Feb. 20)

Example: $y''-3y'-4y=2\sin t$

First guess $Y=A\sin t$

$Y''=-A\sin t$

$Y'=A\cos t$

$-A\sin t-3A\cos t-4A\sin t=2\sin t$

Not possible!

Second guess

$$Y=A\sin t+B\cos t$$

$Y'=A\cos t-B\sin t$

$Y''=-A\sin t-B\cos t$

$$(-A\sin t-B\cos t)-3(A\cos t-B\sin t)-4(A\sin t+B\cos t)$$

$=2\sin t$

$$(-A+3B-4A)\sin t+(-B-3A-4B)\cos t=2\sin t$$

$$3B-5A=2$$
$$-5B-3A=0$$

$A=-\frac{5}{17},\quad B=\frac{3}{17}$

$Y(t)=-\frac{5}{17}\sin t+\frac{3}{17}\cos t.$

Good guess: $Y =$ linear combination of $G(t)$ and its derivatives

Example. $y'' - 3y' - 4y = 4t^2 - 1$

$$Y = A(4t^2 - 1) + B(8t) + C \cdot 8$$
$$= 4At^2 + 8Bt - A + 8C$$
$$= A't^2 + B't + C'$$

Guess $Y = At^2 + Bt + C$

$Y'(t) = 2At + B$

$Y''(t) = 2A$

$$Y'' - 3Y' - 4Y = 2A - 3(2At + B) - 4(At^2 + Bt + C)$$
$$= -4At^2 + (-6A - 4B)t + 2A - 3B - 4C$$

$$\begin{cases} -4A = 4 \\ -6A - 4B = 0 \\ 2A - 3B - 4C = -1 \end{cases} \qquad \begin{aligned} A &= -1 \\ B &= \tfrac{3}{2} \\ C &= -\tfrac{11}{8} \end{aligned}$$

$$Y = -t^2 + \frac{3}{2}t - \frac{11}{8}$$

For $G(t) = P_n(t)$ a polynomial of degree n.

Guess $Y(t) = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0$

Summary

I will only ask you to do these:

$G(t)$	$Y(t)$ (not finalized)
$e^{\alpha t}$	$Ae^{\alpha t}$
$\cos\beta t, \sin\beta t$	$A\cos\beta t + B\sin\beta t$
$P_n(t)$	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_0$
$e^{\alpha t}\begin{pmatrix}\cos\beta t\\ \sin\beta t\end{pmatrix}$	$e^{\alpha t}(A\cos\beta t + B\sin\beta t)$
$P_n(t)e^{\alpha t}$	$(A_n t^n + \cdots + A_0)e^{\alpha t}$
$P_n(t)\begin{pmatrix}\cos\beta t\\ \sin\beta t\end{pmatrix}$	$(A_n t^n + \cdots + A_0)\cos\beta t + (B_n t^n + \cdots + B_0)\sin\beta t$
$P_n(t)e^{\alpha t}\begin{pmatrix}\cos\beta t\\ \sin\beta t\end{pmatrix}$	$(A_n t^n + \cdots + A_0)e^{\alpha t}\cos\beta t + (B_n t^n + \cdots + B_0)e^{\alpha t}\sin\beta t$

Examples:

$-8e^t\cos 2t$	$Ae^t\cos 2t + Be^t\sin 2t$
te^{2t}	$(At+B)e^{2t}$
$t^2\cos 3t$	$(A_1t^2+B_1t+C_1)\cos 3t + (A_2t^2+B_2t+C_2)\sin 3t$

Example $y''-3y'-4y = \underbrace{3e^{2t}}_{G_1(t)} \underbrace{+2\sin t}_{G_2(t)} + \underbrace{4t^2-1}_{G_3(t)}$

$L(Y_1) = G_1 \qquad Y_1 = -\frac{1}{2}e^{2t}$ (previous lecture)

$L(Y_2) = G_2 \qquad Y_2 = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$

$L(Y_3) = G_3 \qquad Y_3 = -t^2 + \frac{3}{2}t - \frac{11}{8}$

$$L(Y_1+Y_2+Y_3) = G_1+G_2+G_3$$

$$Y = Y_1+Y_2+Y_3 = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t - t^2 + \frac{3}{2}t - \frac{11}{8}$$

Example: $y''-3y'-4y = 2e^{-t}$

Guess $Y = Ae^{-t}$

$Y' = -Ae^{-t}$

$Y'' = Ae^{-t}$

$$Y''-3Y'-4Y = \underbrace{Ae^{-t}+3Ae^{-t}-4Ae^{-t}}_{=0} = 2e^{-t}$$

Notice: e^{-t} is a solution to the homogeneous eqn.

Second guess $Y(t) = A \cdot t e^{-t}$

multiply by t

$$Y'(t) = Ae^{-t} - Ate^{-t}$$

$$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$$

$$Y'' - 3Y' - 4Y$$

$$= (-2Ae^{-t} + Ate^{-t}) - 3(Ae^{-t} - Ate^{-t}) - 4Ate^{-t}$$

$$= (-2A - 3A)e^{-t} + (A + 3A - 4A)te^{-t}$$

$$= -5Ae^{-t} = 2e^{-t}$$

Just need $-5A = 2 \Rightarrow A = -\frac{2}{5}$

$$Y = -\frac{2}{5}te^{-t}$$

Sometimes, need to multiply by t^2. (Never more than it for second order D.E.)

Some explanation on multiplication by t.

method of undetermined coefficients for 1st order D.E

① $v' + v = 2$
special solution $V = 2$

② $v' = 2$
special solution $V = 2 \cdot t$

$$y' + y = 2e^{-t}$$

e^{-t} is a soln to homogeneous eqn
e^{t} is the integrating factor

$$e^t y' + e^t y = 2$$

$$(e^t y)' = 2$$

$$v = e^t y, \quad v' = 2.$$

a special soln: $V = 2t$

$$Y = 2te^{-t}$$

Example: $y'' + 4y = \cos 3t - \sin 2t + t^2 - 1$

$L(y) = y'' + 4y$

$L(y_h) = 0$, $\quad y_h = C_1 \cos 2t + C_2 \sin 2t$

soln to homogeneous eqn.

$L(Y_1) = \cos 3t$

$Y_1 = A \cos 3t + B \sin 3t$

$L(Y_2) = -\sin 2t$

$Y_2 = (C \cos 2t + D \sin 2t) \cdot t$

$L(Y_3) = t^2 - 1$

$Y_3 = Et^2 + Ft + G$

Forced Undamped Harmonic oscillator

$$my'' + ky = F(t)$$

external force

We are especially interested in the case

$F = F_0 \cos \omega t$ (periodic driving force)

$$my'' + ky = F_0 \cos \omega t$$

Solution to the homogeneous eqn.

$$C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

$\omega_0 = \sqrt{\frac{k}{m}}$ (Assume $\omega \neq \omega_0$ for now)

Find a special soln to inhomogeneous eqn

$$Y(t) = A \cos \omega t + B \sin \omega t$$

$$Y''(t) = -\omega^2 Y(t)$$

$$m(-\omega^2 Y(t)) + kY(t) = F_0 \cos \omega t$$

$$(k-w^2m)(A\cos wt + B\sin wt) = F_0 \cos wt$$

$$B = 0, \qquad A = \frac{F_0}{k-w^2m} = \frac{F_0}{m(w_0^2-w^2)}$$

$$Y(t) = \frac{F_0}{m(w_0^2-w^2)} \cos wt$$