Lecture 17 (Feb. 22)

Forced Undamped Harmonic oscillator

$$my'' + ky = F(t)$$

external force

We are especially interested in the case

$F = F_0 \cos \omega t$ (periodic driving force)

$$my'' + ky = F_0 \cos \omega t$$

Solution to the homogeneous eqn.

$$C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

$\omega_0 = \sqrt{\frac{k}{m}}$ (Assume $\omega \neq \omega_0$ for now)

Find a special soln to inhomogeneous eqn

$$Y(t) = A \cos \omega t$$

(We don't include a sine function because no first order derivative in DE)

$$Y''(t) = -\omega^2 A \cos \omega t$$

$$-m\omega^2 A \cos \omega t + k A \cos \omega t = F_0 \cos \omega t$$

$$(k-w^2m)A\cos wt = F_0\cos wt$$

$$A = \frac{F_0}{k-w^2m} = \frac{F_0}{m(w_0^2-w^2)} \qquad (k=mw_0^2)$$

$$y(t)=\frac{F_0}{m(w_0^2-w^2)}\cos wt$$

$$y = C_1\cos w_0 t + C_2 \sin w_0 t + \frac{F_0}{m(w_0^2-w^2)}\cos wt$$

If $y(0)=0$, $y'(0)=0$

$$C_1 + \frac{F_0}{m(w_0^2-w^2)}=0$$

$$C_2 w_0 = 0$$

$$C_1 = -\frac{F_0}{m(w_0^2-w^2)}, \quad C_2=0$$

$$y=\frac{F_0}{m(w_0^2-w^2)}(\cos wt - \cos w_0 t)$$

$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$= -\frac{2F_0}{m(w_0^2-w^2)}\sin\frac{w+w_0}{2}t\,\sin\frac{w-w_0}{2}t$$

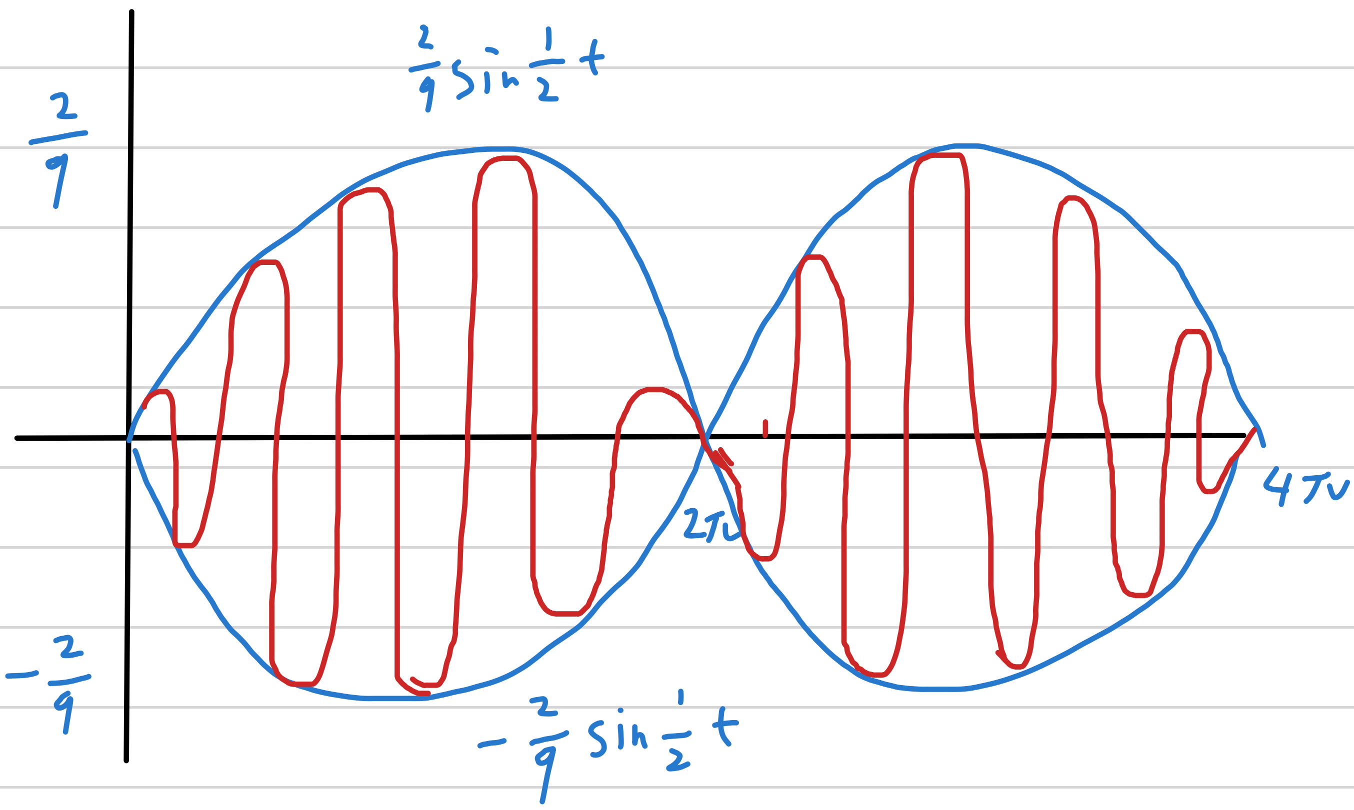
$$=\frac{2F_0}{m(w^2-w_0^2)}\sin\frac{w-w_0}{2}t\,\underbrace{\sin\frac{w+w_0}{2}t}_{\text{rapid oscillating}}$$

amplitude: $\left|\frac{2F_0}{m(w^2-w_0^2)}\sin\frac{w-w_0}{2}t\right|$

Example: $y'' + 16y = \cos 5t \qquad y(0),\ y'(0) = 0$

$$y(t) = \frac{2}{9}\sin\frac{1}{2}t \sin\frac{9}{2}t$$

$\sin\frac{9}{2}t$ has more rapid oscillation than $\sin\frac{1}{2}t$



amplitude $\quad \frac{2}{9}\left|\sin\frac{1}{2}t\right|$: periodic variation

This type of motion is called <u>beats</u>

What happens when $\omega = \omega_0$?

$$y = \frac{2F_0}{m(w^2-w_0^2)} \sin\frac{w-w_0}{2}t \sin\frac{w+w_0}{2}t$$

$$= \frac{2F_0}{m} \frac{\sin\frac{w-w_0}{2}t}{w-w_0} \frac{\sin\frac{w+w_0}{2}t}{w+w_0}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

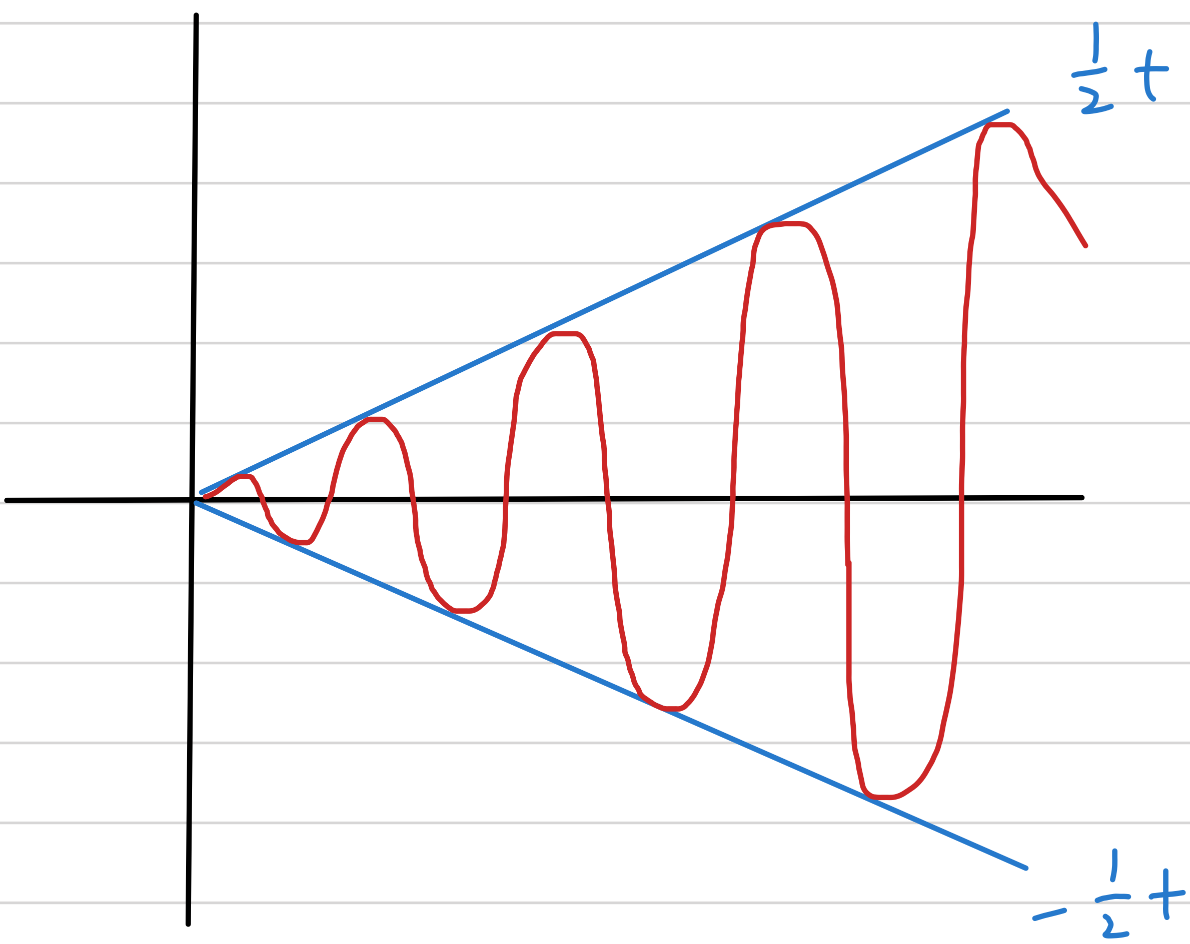
As $w \to w_0$

$$\to \frac{2F_0}{m} \frac{t}{2} \frac{\sin w_0 t}{2w_0}$$

$$= \underbrace{\frac{F_0}{2mw_0}t}_{\text{amplitude (goes to infinity as } t\to\infty)} \sin w_0 t$$

Example $y'' + y = \cos t$, $y(0)=0$, $y'(0)=0$

$$y = \frac{1}{2}t \sin t$$



The amplitude of oscillation blows up If the frequency of the driving force is exactly the same with the natural frequency of the harmonic oscillator

This is called "resonance"

Forced Damped Harmonic Oscillator.

$$m y'' + \gamma y' + k y = F_0 \cos \omega t$$

general solutions

$$y = \underbrace{C_1 y_1 + C_2 y_2}_{y_c} + \underbrace{A \cos \omega t + B \sin \omega t}_{Y}$$

roots of the characteristic eqns r_1, r_2

$$m r^2 + \gamma r + k = 0$$

$\boxed{\gamma > 0}$ r_1, r_2 real and negative

Or complex with negative real part

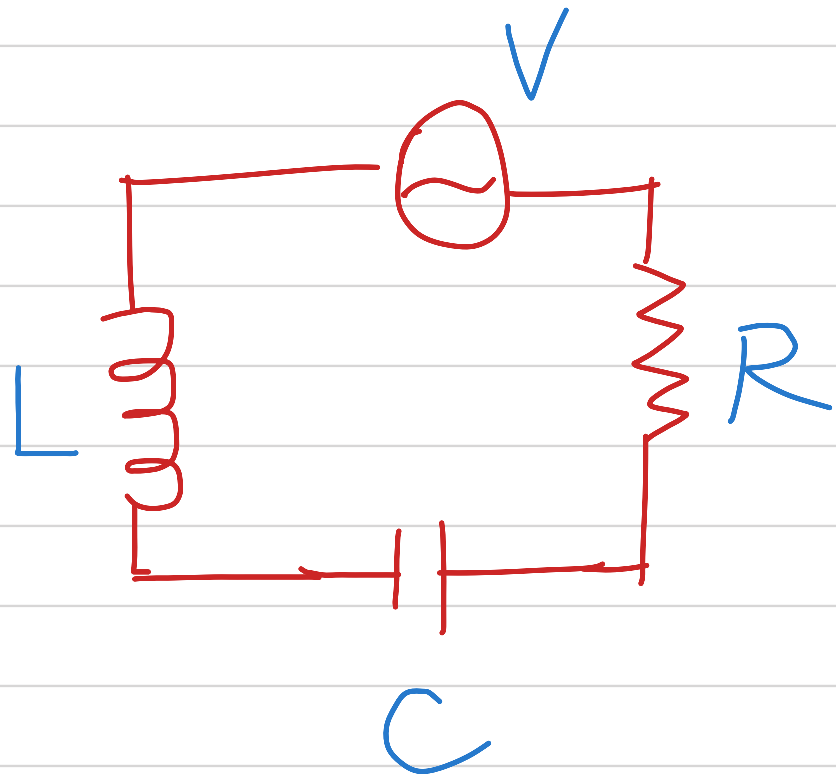
$$y_c(t) \to 0 \quad \text{as } t \to \infty$$

y_c is called the "transient soln"

Y is called the "steady state solution"

or "forced response"

Electrical vibration



q : charge on capacitor

I : electrical current

$$V_C + V_R + V_L = V$$

$$V_C = \frac{q}{C}$$

$$V_R = RI = R\frac{dq}{dt}$$

$$I = \frac{dq}{dt}$$

$$V_L = L\frac{dI}{dt} = L\frac{d^2q}{dt^2}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

$$q(t_0) = q_0 \qquad q'(t_0) = I(t_0) = I_0$$

D.E. for I

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = V'(t)$$

$$I(t_0) = I_0 \; , \; I'(t_0) = \frac{d^2q}{dt^2}(t_0)$$

$$= \frac{1}{L}\left(V(t_0) - \frac{1}{C}Q(t_0) - R\,I(t_0)\right)$$

General procedure of method of undetermined coefficients

$$ay'' + by' + cy = G(t)$$

$$G(t) = P_n(t)\, e^{\alpha t} \begin{pmatrix} \cos\beta t \\ \sin\beta t \end{pmatrix}$$

① Find homogeneous solutions

$$C_1 y_1 + C_2 y_2$$

② Seek special soln Y as a linear combination of G and its derivatives

③ Ask: Are there any terms in Y solutions to the homogeneous eqn?

No → go to step ④

Yes → multiply Y by t, repeat step ③

④ Insert Y into DE and determine the coefficients.

Example: $y'' + y = \cos t$

①. $C_1 \cos t + C_2 \sin t$

② Seek particular soln

$$Y = A\cos t + B\sin t$$

③ $\boxed{\text{Yes}}$ cost, sint are solns to the homogeneous eqn

$$Y = At\cos t + Bt\sin t$$

③ Repeated . $\boxed{\text{No}}$.

④. $Y' = A\cos t - At\sin t + B\sin t + Bt\cos t$

$Y'' = -2A\sin t - At\cos t + 2B\cos t - Bt\sin t$

$Y'' + Y' = -2A\sin t + 2B\cos t = \cos t$

$A = 0, \quad B = \frac{1}{2}$

$$\boxed{Y = \frac{1}{2}t\sin t}$$