Lecture 18 (Feb. 25)

Review of 2nd order D.E.

Linear eqn:

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

Initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Constant coefficients eqns

$$ay'' + by' + cy = G(t)$$

$G(t) = 0$: homogeneous eqn

characteristic eqns

$$ar^2 + br + c = 0$$

$b^2 - 4ac > 0$, two real roots r_1, r_2

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$b^2 - 4ac < 0$, two complex roots $\lambda \pm i\mu$

$$y = e^{\lambda t}(C_1 \cos\mu t + C_2 \sin\mu t)$$

$b^2 - 4ac = 0$. one real root r

$$y = e^{rt}(C_1 + C_2 t)$$

$G(t) \neq 0$, inhomogeneous eqn.

$$y = \underbrace{C_1 y_1 + C_2 y_2}_{} + \underbrace{Y}_{\text{special solution}}$$

To find Y: method of undetermined coefficients

$G(t)$	$Y(t)$
$P_n(t)$	$A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0$
$e^{\alpha t}$	$A e^{\alpha t}$
$\cos\beta t$ or $\sin\beta t$	$A\cos\beta t + B\sin\beta t$

If some term in Y is solution to homogeneous eqn, $\boxed{\text{multiply by } t}$

Harmonic oscillator

$$my'' + \gamma y' + ky = F(t)$$

m: mass

γ: damping coefficient

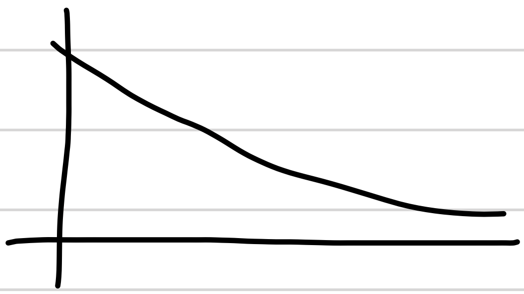
k: spring constant

F: driving force.

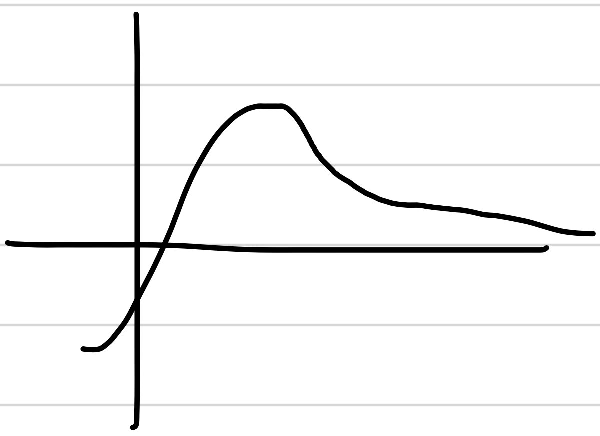
$m > 0, \quad k > 0, \quad \gamma \geq 0$ $\left(\begin{array}{ll} \gamma = 0 & \text{undamped} \\ \gamma > 0 & \text{damped} \end{array} \right)$

Unforced Harmonic oscillator

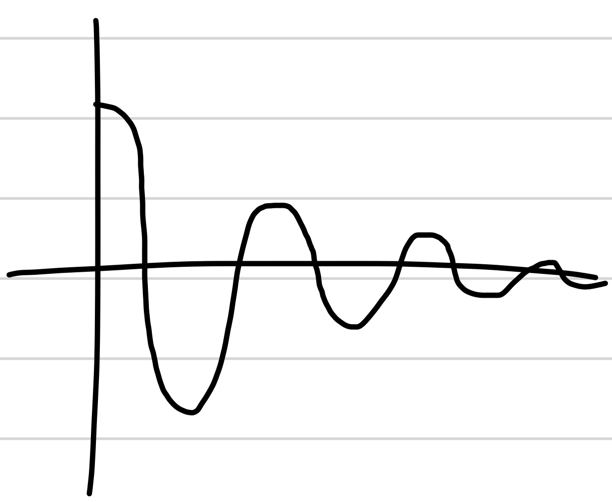
$\gamma^2 - 4mk > 0$, over-damped.



$\gamma^2 - 4mk = 0$, critically damped.



$\gamma^2 - 4mk < 0$. under damped



For the under-damped case : $\boxed{\lambda \pm i\mu}$

$$w = \mu = \frac{\sqrt{4km - \gamma^2}}{2m}$$: quasi-frequency

$$\lambda = -\frac{\gamma}{2m}$$

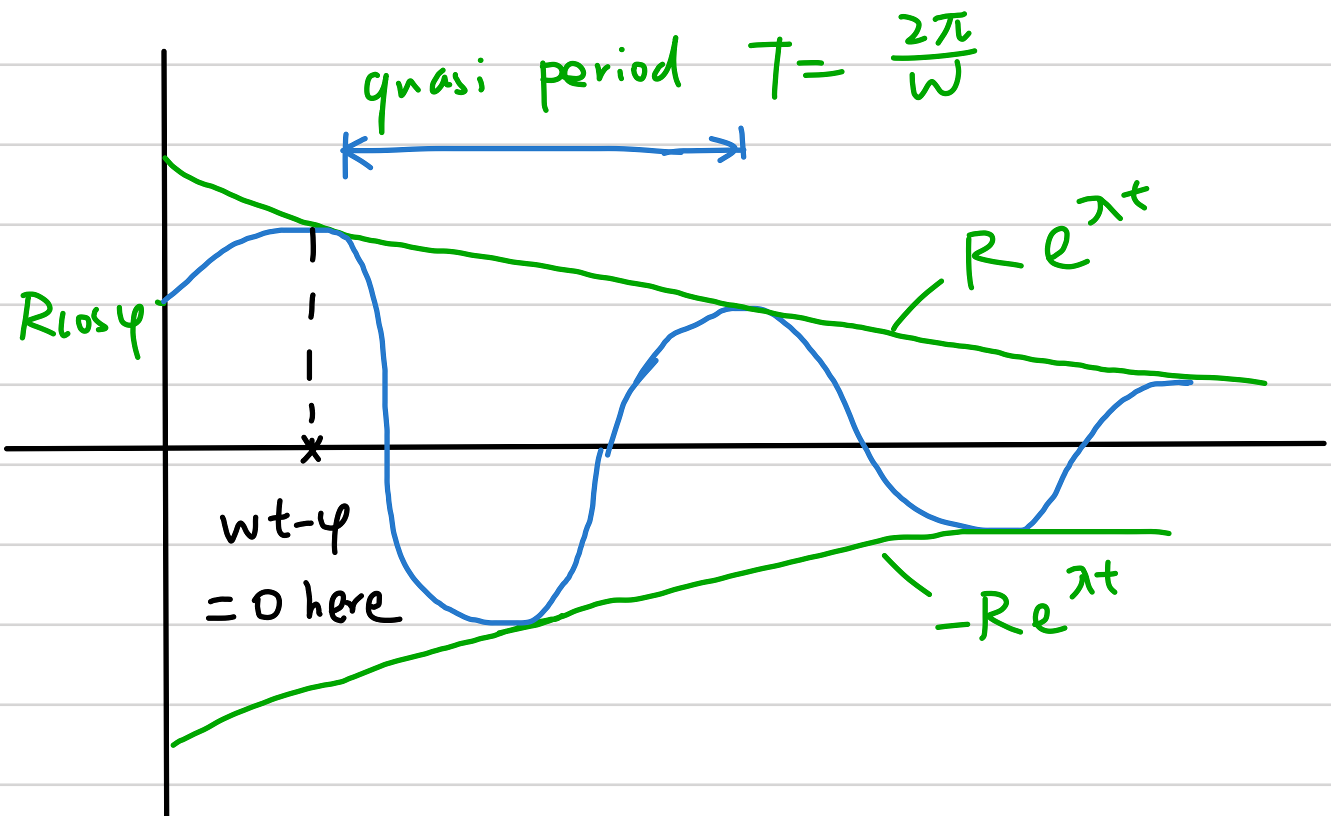
quasi-period : $T = \frac{2\pi}{\omega}$

solution written in

$$y = \underbrace{Re^{\lambda t}}_{\text{amplitude}} \cos(\omega t - \varphi)$$

phase

I used the convention $\varphi \in [0, 2\pi)$ in class
you don't need to use this convention



$\gamma = 0$, $\omega_0 = \sqrt{\frac{k}{m}}$: natural frequency

$T_0 = \frac{2\pi}{\omega_0}$: period

$$y = R\cos(\omega_0 t - \varphi)$$

Forced Undamped Harmonic oscillator

$$my'' + ky = F_0 \cos wt \quad , y(0) = 0, \quad y'(0) = 0$$

If $w \neq w_0$

$$y = \frac{2F_0}{m(w^2 - w_0^2)} \sin \frac{w - w_0}{2} t \sin \frac{w + w_0}{2} t$$

beats : amplitude $\left| \frac{2F_0}{m(w^2 - w_0^2)} \sin \frac{w - w_0}{2} t \right|$

If $w = w_0$

$$y = \frac{F_0}{2mw_0} t \cos w_0 t$$

resonance : amplitude $\frac{F_0}{2mw_0} t$

Forced Damped Harmonic Oscillator

$$my'' + ry' + ky = F_0 \cos wt$$

$$y = \underbrace{C_1 y_1 + C_2 y_2}_{\substack{\text{transient} \\ \text{solution}}} + \underbrace{A \cos wt + B \sin wt}_{\substack{\text{steady state} \\ \text{solution}}}$$