# Lecture 2. (Jan 9.)

variables $x, y$ constants: $a, b, c$

Linear function $f(x,y) = ax + by - c$

only constant multiples of $x, y$

nonlinear terms. $xy$, $x^2$, $\sin x$

Linear function of $u(t), v(t)$

$$F(u(t), v(t), t) = a(t)u(t) + b(t)v(t) - c(t)$$

in front of $u, v$ — only fcns of $t$.

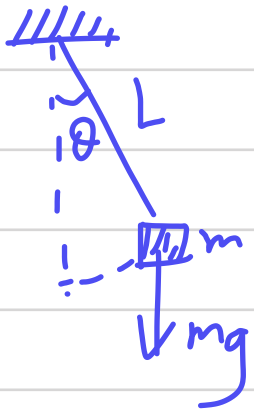
$a(t), b(t), c(t)$ not necessarily linear in $t$

nonlinear terms $u^2(t)$, $u(t)v(t)$, $\sin u(t)$

$(\sin t)u$ linear!

Example of a nonlinear equation

Differential equation for $\theta(t)$




$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$\sin\theta \sim \theta$ for small $\theta$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \quad \text{(linearization)}$$

Linear or nonlinear?

① $u\frac{du}{dt} = 0$

② $\frac{d^2u}{dt^2} - \sin t\,\frac{du}{dt} + \cos t\; u = 0$

Solve the eqn.

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

$$\frac{dv}{dt} = \frac{49 - v}{5}$$

$$\frac{dv}{(v-49)dt} = -\frac{1}{5}$$

$$\ln|v-49| = -\frac{t}{5} + C$$

$$|v-49| = e^{-\frac{t}{5}+C}$$

$$= e^C \cdot e^{-\frac{t}{5}}$$

$$\frac{d}{dx}\ln|x-a| = \frac{1}{x-a}$$

use chain rule

$$(\ln|v-a|)' = \frac{1}{v-a}\cdot v'$$

$v - 49 = \pm e^{C} e^{-\frac{t}{5}}$ $\quad c = \pm e^{C}$

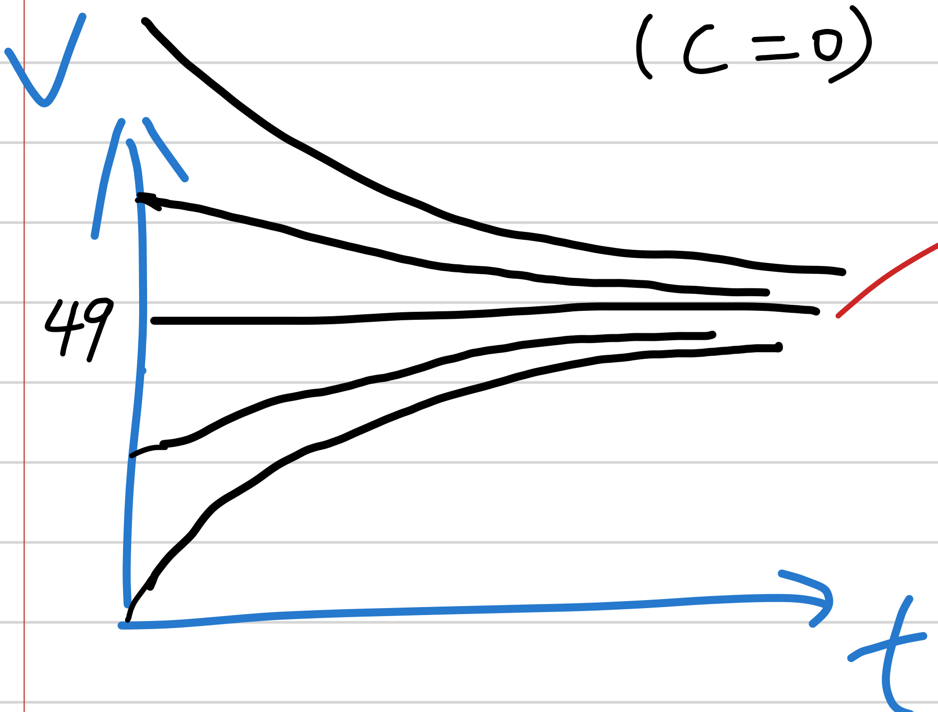
$v(t) = 49 + ce^{-\frac{t}{5}}$

notice $t \to \infty$ $\quad e^{-\frac{t}{5}} \to 0$

$v \to 49$

$v(t) \equiv 49$ is a solution!

$(c = 0)$




equilibrium solution

solution for $v(0) = 0$

$0 = v(0) = 49 + ce^{0}$

$\Rightarrow c = -49$

$v(t) = 49(1 - e^{-t/5})$

for a point $(t_0, v_0)$, there exists a single solution passing through this point $\quad v_0 = v(t_0) = 49 + ce^{-\frac{t_0}{5}}$

Solve for $c$!

Equilibrium solution $v(t)$:

$$\frac{dv}{dt} = 0$$

(stable one: other solns move toward it)

If $v(0) = 49$, $v(t) \equiv 49$.

Solve $\frac{dp}{dt} = 0.5p - 450$

$$\frac{dp}{dt} = \frac{p-900}{2}$$

$$\frac{dp}{(p-900)dt} = \frac{1}{2}$$

$$\ln|p-900| = \frac{t}{2} + C$$

$$p - 900 = \pm e^{\frac{t}{2}+C} = c\,e^{\frac{t}{2}}$$

$$p = 900 + ce^{\frac{t}{2}}$$

$t \to \infty$: $e^{\frac{t}{2}} \to \infty$

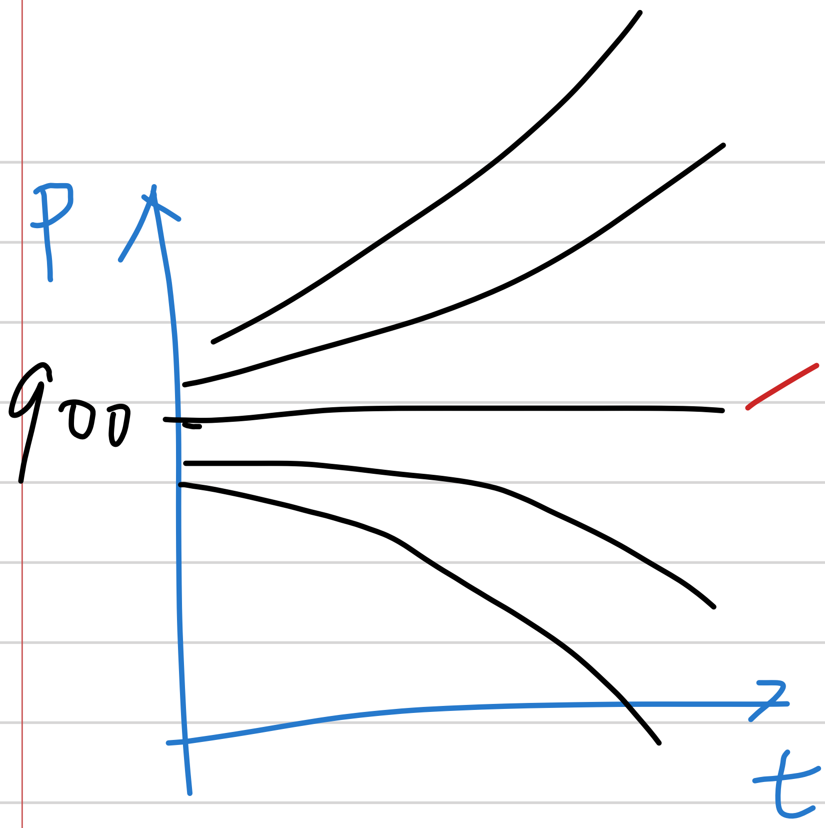
$c > 0$ $\quad p(t) \to +\infty$

$c < 0$ $\quad p(t) \to -\infty$

$c = 0$ $\quad \underline{p(t) \equiv 900}$ equilibrium sol

Remark: $p(t) \to -\infty$ not realistic

population nonnegative




equilibrium (unstable)

$900 + c = p(0)$

$c = p(0) - 900$

$c > 0$ if $p(0) > 900$

$c < 0$ if $p(0) < 900$

Solve $mx'' = -kx$

denote $w^2 = \frac{k}{m}$

$x'' = -w^2 x$

w: natural frequency

solution: $x(t) = \sin wt$

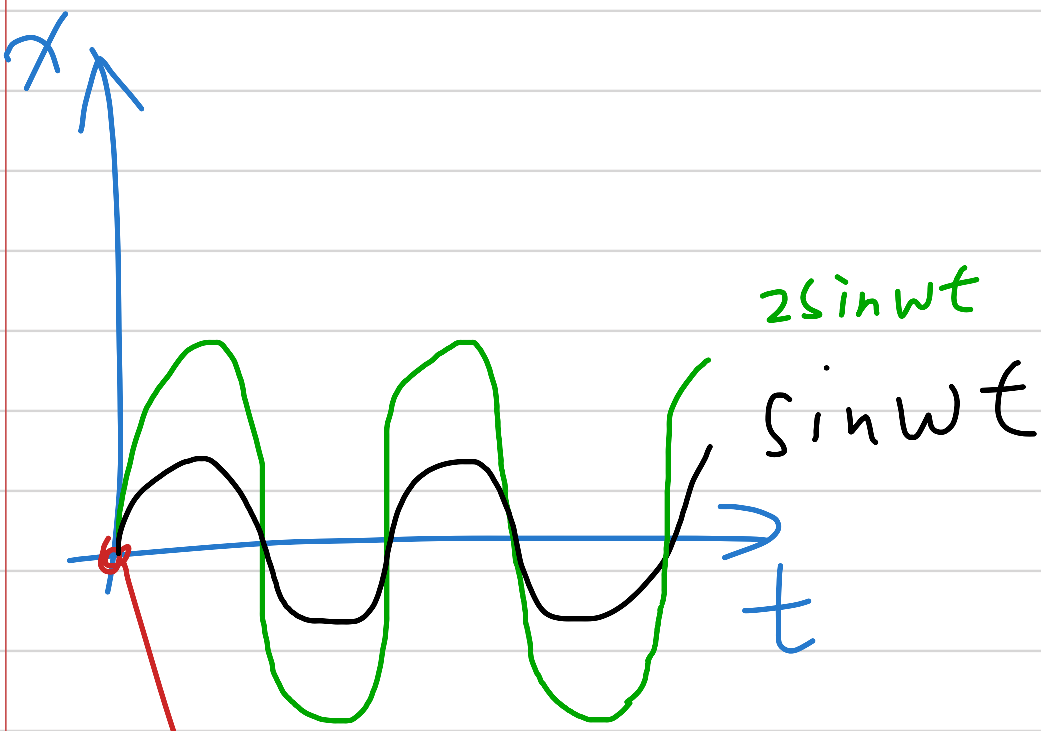
$x'(t) = w\cos wt$

$x''(t) = -w^2 \cos wt = -w^2 x(t)$

$x(t) = C\sin wt$

any constant

$x(t) = C\cos wt$ any constant C.




many solutions through this point

(This is a second order equation!)