Lecture 20. (March 1)

Laplace Transform - a method for solving constant coefficient DE/IVPs

A function $f(t)$ defined on $[0, +\infty)$

Its Laplace Transform is

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Notation: $F(s) = \mathcal{L}\{f(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$

Let us calculate the Laplace Transform for some functions

Example 1. $f(t) = 1$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\,dt \qquad s \neq 0$$

$$= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-st} \Big|_0^A \right]$$

$\lim_{A\to\infty} e^{-sA} = 0$ if $s > 0$

$= \infty$ if $s < 0$

$$= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-sA} - \left(-\frac{1}{s} e^{-s\cdot 0} \right) \right]$$

$$= \frac{1}{s} \qquad s > 0$$

You can always assume s is big enough

Example: $f(t) = e^{-t}$

$$\mathcal{L}\{e^{-t}\} = \int_0^\infty e^{-st} e^{-t} dt$$

$$= \int_0^\infty e^{-(s+1)t} dt$$

$$= -\frac{1}{s+1} e^{-(s+1)t} \Big|_0^\infty$$

$$= -\frac{1}{s+1} \cdot 0 + \frac{1}{s+1} \cdot 1$$

$$= \frac{1}{s+1} \qquad s > -1$$

Example: $f(t) = e^{at}$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \frac{1}{s-a}$$

Why Laplace Transform?

$$\mathcal{L}\{y'\} = \lim_{A\to\infty} \int_0^A e^{-st} y'(t) dt$$

$$= \lim_{A\to\infty} e^{-st} y(t) \Big|_0^A - \int_0^A y(t) d\, e^{-st}$$

$$= \lim_{A\to\infty} \left(\frac{y(A)}{e^{sA}} \right) - y(0) - \int_0^\infty (-s) e^{-st} y(t)\, dt$$

$$= -y(0) + s\int_0^\infty e^{-st} y(t)\,dt$$

$$= s\mathcal{L}\{y\} - y(0)$$

Laplace Transform is linear.

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{cf(t)\} = c\mathcal{L}\{f(t)\}$$

$f(t)$ is uniquely determined by $\mathcal{L}\{f(t)\}$

Now, let us solve the following IVP.

$$\begin{cases} \dfrac{dy}{dt} + 4y = 0 \\ y(0) = 1 \end{cases}$$

$$\mathcal{L}\left\{\frac{dy}{dt} + 4y\right\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\mathcal{L}\{y\} = 0$$ use the linearity

$$sY(s) - 1 + 4Y(s) = 0$$ (algebraic eqn)

$$(s+4)Y(s) = 1$$

$$Y(s) = \frac{1}{s+4} = \mathcal{L}\{e^{-4t}\}$$

$$\Rightarrow y(s) = e^{-4t}$$

$$\mathcal{L}\{y''\} = s\mathcal{L}\{y'\} - y'(0)$$
$$= s[s\mathcal{L}\{y\} - y(0)] - y'(0)$$
$$= s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

$$\mathcal{L}\{y^{(n)}\} = s^n\mathcal{L}\{y\} - s^{n-1}y(0) - \cdots - s\,y^{(n-2)}(0) - y^{(n-1)}(0)$$

IVP for
$y' + ay = f(t)$
$ay'' + by' + cy = g(t)$
etc.

$\mathcal{L}$ ↓ Laplace Transform

$sY + aY = F(s)$
$as^2Y + bsY + cY = G(s)$
(just assume zero initial conditions)

$\xrightarrow{\text{solve algebraic equations}}$ [easy]

$Y(s)$

$\mathcal{L}^{-1}$ ↑ Inverse Laplace Transform

solution
$y(t)$

Example: $\sin at$, $\cos at$

$$\mathcal{L}\{\sin at\} = \mathcal{L}\{(-\tfrac{1}{a}\cos at)'\}$$

$$= -\frac{1}{a}\mathcal{L}\{(\cos at)'\}$$

$$= -\frac{1}{a}\left(s\mathcal{L}\{\cos at\} - 1\right)$$

$$= -\frac{s}{a}\mathcal{L}\{\cos at\} + \frac{1}{a}$$

$$\mathcal{L}\{\cos at\} = \mathcal{L}\{(\tfrac{1}{a}\sin at)'\}$$

$$= \frac{1}{a}\mathcal{L}\{(\sin at)'\}$$

$$= \frac{s}{a} \cdot \mathcal{L}\{\sin at\}$$

$$\mathcal{L}\{\sin at\} = \frac{1}{a} - \frac{s^2}{a^2}\mathcal{L}\{\sin at\}$$

$$\left(1 + \frac{s^2}{a^2}\right)\mathcal{L}\{\sin at\} = \frac{1}{a}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{a}\mathcal{L}\{\sin at\} = \frac{s}{s^2+a^2}$$

Example: $\mathcal{L}\{t\}$ $\qquad 1 = \dot{t}$

$$\mathcal{L}\{1\} = \mathcal{L}\{\dot{t}\}$$
$$= s \cdot \mathcal{L}\{t\} - 0$$
$$\mathcal{L}\{t\} = \frac{1}{s}\mathcal{L}\{1\}$$
$$= \frac{1}{s^2}$$

Example: $f(t) = t^n$ $\qquad (t^n)' = n t^{n-1}$

$$\mathcal{L}\{n t^{n-1}\} = s \mathcal{L}\{t^n\} - 0 \qquad n \geq 1$$
$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$$
$$\mathcal{L}\{t^2\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2}{s^3}$$
$$\mathcal{L}\{t^3\} = \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3}{s} \cdot \frac{2}{s^3} = \frac{3!}{s^4}$$

...

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$