Lecture 21 (March 4)

Property 1: $\mathcal{L}\{e^{at}y(t)\} = Y(s-a)$ (exponential shift formula)

$$\mathcal{L}\{e^{at}y(t)\}$$
$$= \int_0^{+\infty} e^{-st}e^{at}y(t)\,dt$$
$$= \int_0^{+\infty} e^{-(s-a)t}y(t)\,dt$$
$$= Y(s-a)$$

Examples: $\mathcal{L}\{e^{at}\cos wt\}$

we know $\mathcal{L}\{\cos wt\} = \dfrac{s}{s^2+w^2}$

$$\mathcal{L}\{e^{at}\cos wt\} = \frac{s-a}{(s-a)^2+w^2}$$

$\mathcal{L}\{e^{at}\sin wt\}$ $\qquad$ $\mathcal{L}\{\sin wt\} = \dfrac{w}{s^2+w^2}$

$$\mathcal{L}\{e^{at}\sin wt\} = \frac{w}{(s-a)^2+w^2}$$

Property 2. $\mathcal{L}\{t\,y(t)\} = -\dfrac{d}{ds}Y(s) = -\dfrac{d}{ds}\mathcal{L}\{y(t)\}$

$$-\frac{d}{ds}Y(s) = -\frac{d}{ds}\int_0^{+\infty} e^{-st} y(t)\,dt$$
$$= -\int_0^{+\infty} \frac{d}{ds}(e^{-st})\, y(t)\,dt$$
$$= -\int_0^{+\infty} (-te^{-st})\, y(t)\,dt$$
$$= \int_0^{+\infty} te^{-st} y(t)\,dt$$
$$= \mathcal{L}\{ty(t)\}$$

Example: $\mathcal{L}\{te^{at}\}$

$$\mathcal{L}\{te^{at}\} = -\frac{d}{ds}\mathcal{L}\{e^{at}\}$$
$$= -\frac{d}{ds}\left(\frac{1}{s-a}\right)$$
$$= \frac{1}{(s-a)^2}$$

For IVP: $ay'' + by' + cy = f(t)$

$$\xrightarrow{\mathcal{L}} as^2\mathcal{L}\{y\} + bs\mathcal{L}\{y\} + c\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$ + some function of s (comes from the I.C.)

$$(as^2+bs+c)Y(s) = Q(s)$$
$$Y(s) = \frac{Q(s)}{as^2+bs+c}$$

need to find $Y(s) \rightarrow y(t)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Inverse Laplace Transform $\mathcal{L}^{-1}$

We have a formula to compute the Laplace Transform, but there is no formula to compute the Inverse Laplace Transform.

Need to use the table of Laplace Transforms

- $\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$
- $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = \mathcal{L}^{-1}\left\{\text{a shift of } \frac{1}{s^2}\right\}$

use the exponential shift formula $t \rightsquigarrow \frac{1}{s^2}$

$e^{at}y(t) \rightsquigarrow Y(s-a)$ $\qquad e^{4t} \cdot t \rightsquigarrow \frac{1}{(s-4)^2}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s-a)\} = e^{at}\mathcal{L}^{-1}\{Y(s)\}$

apply to $Y(s) = \frac{1}{s^2}$, $a = 4$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{4t} \cdot t$$

Apply $\mathcal{L}^{-1}\{-\frac{d}{ds}Y(s)\}$

$= t \cdot \mathcal{L}^{-1}\{Y(s)\}$

to $Y(s) = \frac{1}{s-4}$

Or alternatively, notice

$$\frac{d}{ds}\left(\frac{1}{s-4}\right) = -\frac{1}{(s-4)^2}$$

$$\frac{1}{(s-4)^2} = -\frac{d}{ds}\left(\frac{1}{s-4}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = t\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = te^{4t}$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-8}\right\}$

$$\frac{1}{s^2+2s-8} = \frac{1}{(s+4)(s-2)}$$

$$= \frac{A}{s+4} + \frac{B}{s-2}$$

$$= \frac{A(s-2)+B(s+4)}{(s+4)(s-2)}$$

$$= \frac{(A+B)s - 2A + 4B}{(s+4)(s-2)}$$

$A+B = 0$, $-2A+4B = 1$

$B = \frac{1}{6}$ $\quad A = -\frac{1}{6}$

$$\frac{1}{(s-2)(s+4)} = \frac{1/6}{s-2} - \frac{1/6}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+4)}\right\} = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= \frac{1}{6}e^{2t} - \frac{1}{6}e^{-4t}$$

- $\mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+9}\right\}$

$$\frac{3s+4}{s^2+9} = 3\frac{s}{s^2+3^2} + \frac{4}{3}\cdot\frac{3}{s^2+3^2}$$

$$\mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+9}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= 3\cdot\cos 3t + \frac{4}{3}\sin 3t$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$

$$\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4}$$ = a shift of $\frac{1}{s^2+4}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{e^{-t}}{2} \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \frac{e^{-t}}{2} \sin 2t$$

- $\mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+2s+5}\right\}$

$$\frac{3s+4}{s^2+2s+5} = \frac{3s+4}{(s+1)^2+4}$$

$$= \frac{3(s+1)+1}{(s+1)^2+4}$$

$$= \text{shift of } \frac{3s+1}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+2s+5}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{3\cdot\frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}\right\}$$

$$= e^{-t}\left[3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$$

$$= e^{-t}\left(3\cos 2t + \frac{1}{2}\sin 2t\right)$$