

Lecture 22 (March 6)

Solve the IVP using Laplace Transform

$$y'' + 2y' + 10y = 0$$

$$y(0) = 1, \quad y'(0) = 2$$

$$\mathcal{L}\{y'' + 2y' + 10y\} = 0$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 0$$

$$[s^2 Y(s) - s y(0) - y'(0)] + 2[s Y(s) - y(0)] + 10 Y(s) = 0$$

$$s^2 Y(s) - s - 2 + 2(s Y(s) - 1) + 10 Y(s) = 0$$

$$(s^2 + 2s + 10) Y(s) - s - 4 = 0$$

$$Y(s) = \frac{s + 4}{s^2 + 2s + 10}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{s+4}{s^2+2s+10} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{(s+1)+3}{(s+1)^2+9} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{s+1}{(s+1)^2+9} \right\} + \mathcal{L}^{-1} \left\{ \frac{3}{(s+1)^2+9} \right\}$$

$$= e^{-t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} + e^{-t} \mathcal{L}^{-1} \left\{ \frac{3}{s^2+9} \right\}$$

$$= e^{-t} \cos 3t + e^{-t} \sin 3t$$

Example. $y'' + y = \sin 2t$
 $y(0) = 2, \quad y'(0) = 1$

$$\mathcal{L} \{ y'' + y \} = \mathcal{L} \{ \sin 2t \}$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$$

$$(s^2+1)Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

Partial fraction decomposition for $\frac{2}{(s^2+1)(s^2+4)}$

$$\left[\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \right] \times (s^2+1)(s^2+4)$$

$$2 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$= As^3 + 4As + Bs^2 + 4B$$

$$+ Cs^3 + Ds^2 + Cs + D$$

$$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + 4B+D$$

$A+C=0,$	$B+D=0$
$4A+C=0$	$4B+D=2$

$$A=0, C=0$$

$$B = \frac{2}{3}, D = -\frac{2}{3}$$

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{2}{3} \cdot \frac{1}{s^2+1} - \frac{2}{3} \cdot \frac{1}{s^2+4}$$

$$Y(s) = \frac{2}{3} \frac{1}{s^2+1} - \frac{2}{3} \frac{1}{s^2+4} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{5}{3} \frac{1}{s^2+1} - \frac{1}{3} \frac{2}{s^2+4} + 2 \cdot \frac{s}{s^2+1}$$

$\downarrow \mathcal{L}^{-1}$

$$y(t) = \frac{5}{3} \sin t - \frac{1}{3} \sin 2t + 2 \cdot \cos t$$

Example: $y'' + y = e^{-2t}$

$$y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{e^{-2t}\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \frac{1}{s+2}$$

$$s^2 Y(s) + Y(s) = \frac{1}{s+2}$$

$$(s^2+1) Y(s) = \frac{1}{s+2}$$

$$Y(s) = \frac{1}{(s+2)(s^2+1)}$$

Need to find $y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\}$

- partial fraction decomposition

$$\left[\frac{1}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1} \right] (s+2)(s^2+1)$$

$$\begin{aligned} 1 &= A(s^2+1) + (Bs+C)(s+2) \\ &= As^2 + A + Bs^2 + Cs + 2Bs + 2C \\ &= (A+B)s^2 + (C+2B)s + A+2C \end{aligned}$$

$$\left. \begin{array}{l} A+B=0 \quad \textcircled{1} \\ C+2B=0 \quad \textcircled{2} \\ A+2C=1 \quad \textcircled{3} \end{array} \right\}$$

Use $\textcircled{1}$ $\textcircled{2}$ $\textcircled{3}$ to eliminate A ($\textcircled{3}-\textcircled{1}$)

$$2C - B = 1$$

together with $C+2B=0$ $\textcircled{2} \Rightarrow C = -2B$

$$-4B - B = 1$$

$$B = -\frac{1}{5}$$

$$A = -B = \frac{1}{5}, \quad C = -2B = \frac{2}{5}$$

$$\frac{1}{(s+2)(s^2+1)} = \frac{1/5}{s+2} + \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2+1}$$

$$= \frac{1}{5} \cdot \frac{1}{s+2} - \frac{1}{5} \frac{s}{s^2+1} + \frac{2}{5} \frac{1}{s^2+1}$$

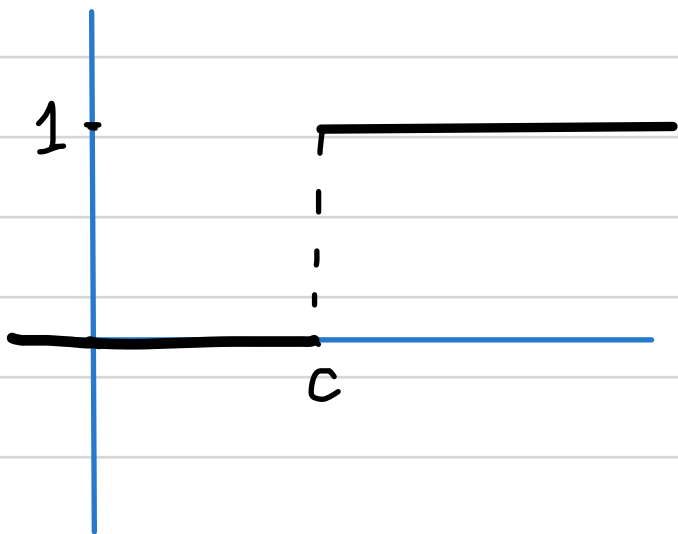
↓ \mathcal{L}^{-1}

$$y(t) = \frac{1}{5} e^{-2t} - \frac{1}{5} \cos t + \frac{2}{5} \sin t$$

We can solve inhomogeneous eqn with piecewise continuous right hand side $f(t)$ using Laplace Transform

special soln
homogeneous solution

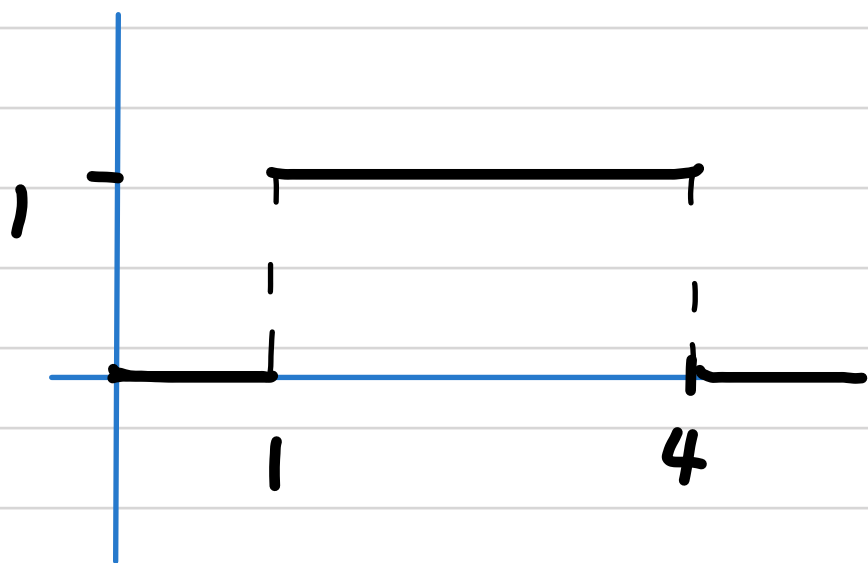
Heaviside function (unit step function)



$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t > c \end{cases}$$

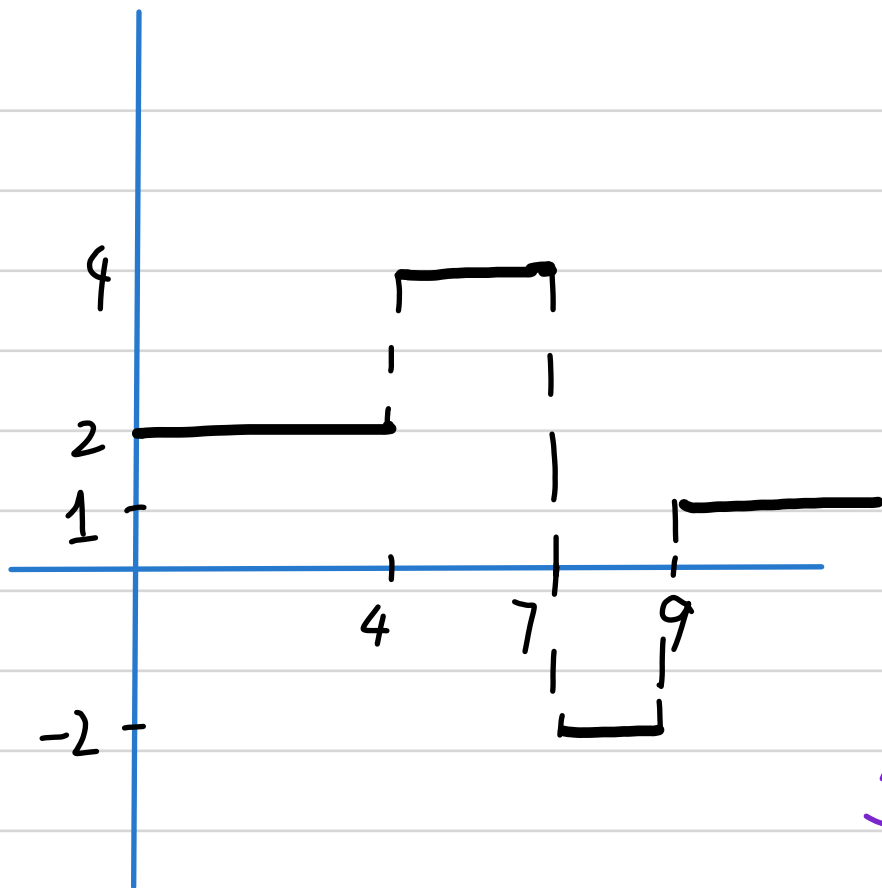
$$\begin{aligned}
 \mathcal{L}\{u_c(t)\} &= \int_0^{+\infty} e^{-st} u_c(t) dt \\
 &= \int_c^{+\infty} e^{-st} \cdot 1 dt \\
 &= -\frac{1}{s} e^{-st} \Big|_c^{+\infty} \\
 &= 0 - \left(-\frac{1}{s} e^{-s \cdot c}\right) \quad s > 0 \\
 &= \frac{1}{s} e^{-cs}
 \end{aligned}$$

Example: $h(t) = u_1(t) - u_4(t)$



$$\begin{aligned}
 \mathcal{L}\{h(t)\} &= \mathcal{L}\{u_1(t)\} - \mathcal{L}\{u_4(t)\} \\
 &= \frac{e^{-s}}{s} - \frac{e^{-4s}}{s}
 \end{aligned}$$

$$\text{Example: } f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 4 & 4 < t < 7 \\ -2 & 7 < t < 9 \\ 1 & t > 9 \end{cases}$$



start with
 $f_1(t) = 2$

jump of 2 at
 $t = 4$

$$f_2(t) = f_1(t) + 2u_4(t)$$

$$= 2 + 2u_4(t)$$

jump of -6 at
 $t = 7$

$$f_3(t) = f_2(t) - 6u_7(t)$$

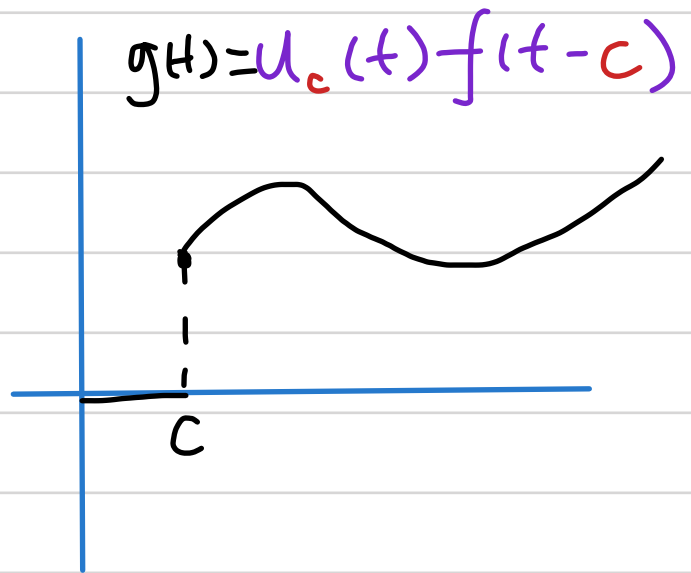
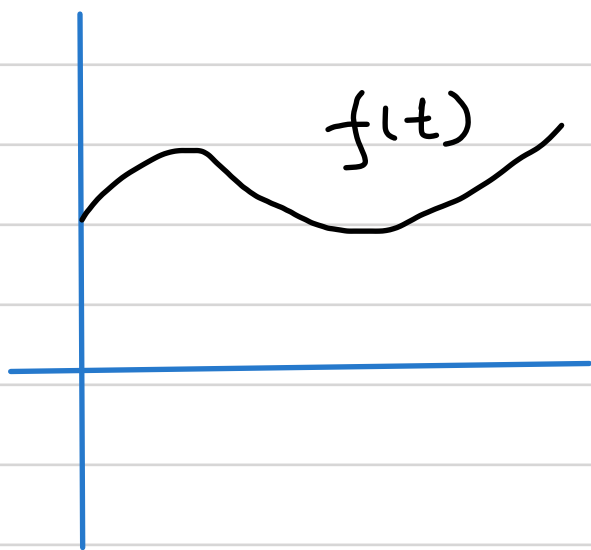
$$= 2 + 2u_4(t) - 6u_7(t)$$

jump of 3 at
 $t = 9$

$$f_4(t) = f_3(t) + 3u_9(t)$$

$$= 2 + 2u_4(t) - 6u_7(t) + 3u_9(t)$$

$$f(t) = 2 + 2u_4(t) - 6u_7(t) + 3u_9(t)$$



$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$$

$$\mathcal{L} \{ u_c(t) f(t-c) \} = \int_0^{+\infty} e^{-st} u_c(t) f(t-c) dt$$

$$= \int_c^{+\infty} e^{-st} f(t-c) dt$$

$$(t-c = \xi) = \int_0^{+\infty} e^{-s(\xi+c)} f(\xi) d\xi$$

$$= e^{-sc} \int_0^{+\infty} e^{-s\xi} f(\xi) d\xi$$

$$= e^{-cs} \mathcal{L} \{ f(t) \}$$