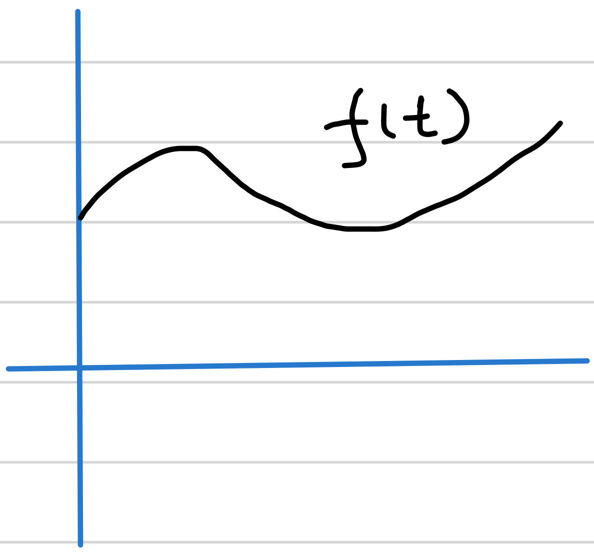
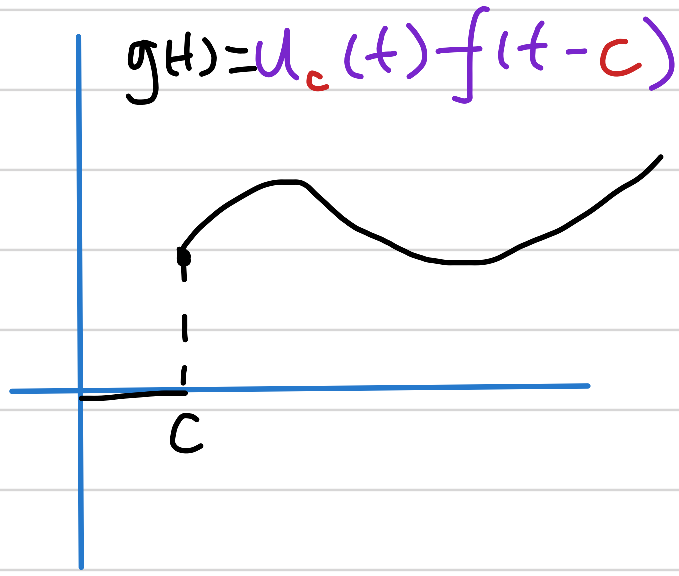
# Lecture 23 (March 8)

$f(t)$ delayed by $c$




$g(t) = u_c(t) f(t-c)$




$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$$

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^{+\infty} e^{-st} u_c(t) f(t-c)\,dt$$

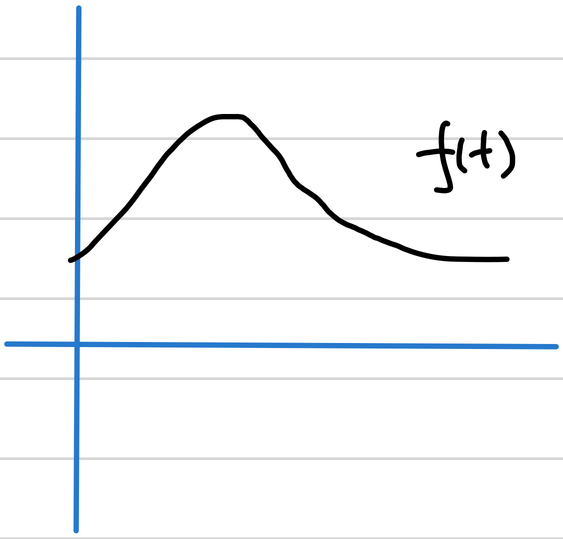
$$= \int_c^{\infty} e^{-st} f(t-c)\,dt$$

$$(t-c=\xi) \quad = \int_0^{+\infty} e^{-s(\xi+c)} f(\xi)\,d\xi$$

$$= e^{-sc} \int_0^{+\infty} e^{-s\xi} f(\xi)\,d\xi$$

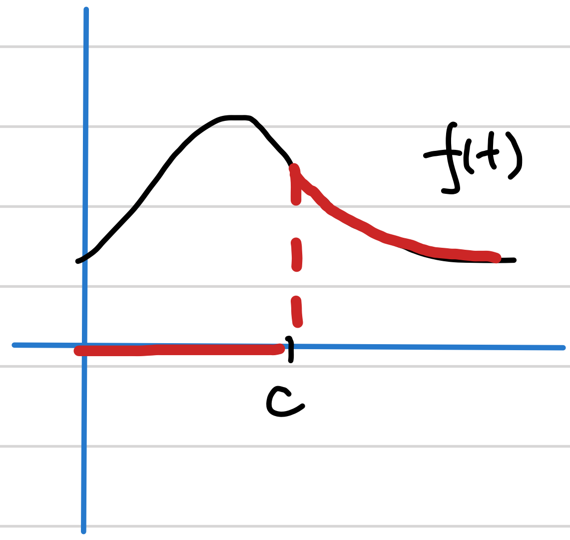
$$= e^{-cs} \mathcal{L}\{f(t)\}$$

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t) f(t-c)$$



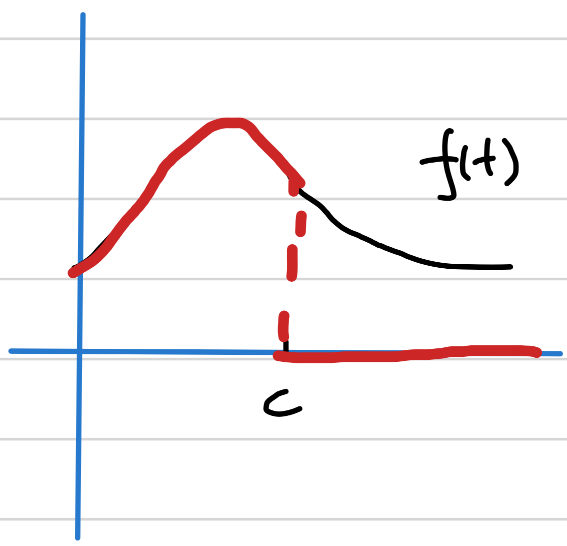

$$g_1(t) = f(t)\,u_c(t)$$

$$= \begin{cases} 0 & t < c \\ f(t) & t > c \end{cases}$$




$$g_2(t) = f(t)\,(1 - u_c(t))$$

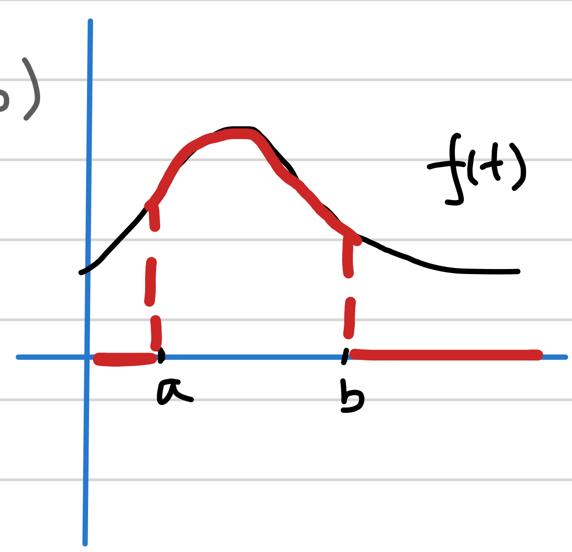
$$= \begin{cases} f(t) & t < c \\ 0 & t > c \end{cases}$$




$(a < b)$

$$g_3(t) = f(t)\,(u_a(t) - u_b(t))$$

$$= \begin{cases} 0 & 0 \le t < a \\ f(t) & a < t < b \\ 0 & t > b \end{cases}$$



Example. $f(t) = \begin{cases} t & 0 \le t < 1 \\ e^{(t-1)} & t > 1 \end{cases}$

Approach 1 $f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & t > 1 \end{cases}$

$$+ \begin{cases} 0 & 0 \le t < 1 \\ e^{(t-1)} & t > 1 \end{cases}$$

$$= t \cdot (1 - u_1(t)) + e^{(t-1)} u_1(t)$$

$$= t + u_1(t)(e^{(t-1)} - t)$$

Approach 2. Start with $f_1(t) = t$

from $t = 1$, $t$ becomes $e^{(t-1)}$

$$f_2(t) = t + u_1(t)[e^{(t-1)} - t]$$

Let's calculate the Laplace transforms of some discontinuous functions.

Example. $\mathcal{L}\{\underline{e^{(t-1)} u_1(t)}\} \longrightarrow$ $e^t$ delayed by 1.

$$= e^{-s} \mathcal{L}\{e^t\}$$

$$= e^{-s} \cdot \frac{1}{s-1}$$

Example: $\mathcal{L}\{e^t u_1(t)\}$ $\longrightarrow$ $u_1(t)\, e^{(t-1)+1} = e^{t+1}$ delayed by 1

$= e^{-s}\mathcal{L}\{e^{t+1}\}$

$= e^{-s}\cdot e\,\mathcal{L}\{e^{t}\}$

$= e^{-s+1}\dfrac{1}{s-1}$

Example. $f(t) = \begin{cases} \sin t & 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t-\frac{\pi}{4}) & t > \frac{\pi}{4} \end{cases}$

$f(t) = \sin t + \underline{u_{\frac{\pi}{4}}(t)\cos(t-\tfrac{\pi}{4})}$ $\quad$ cos t delayed by $\frac{\pi}{4}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{\underline{u_{\frac{\pi}{4}}(t)\cos(t-\tfrac{\pi}{4})}\}$

$= \mathcal{L}\{\sin t\} + e^{-\frac{\pi}{4}\cdot s}\mathcal{L}\{\cos t\}$

$= \dfrac{1}{s^2+1} + e^{-\pi s/4}\cdot\dfrac{s}{s^2+1}$

Example: $\mathcal{L}\{u_4(t)\sin 2t\}$ $\quad$ time delay of $\boxed{f = \sin 2(t+4)}$

$= \mathcal{L}\{u_4(t)\sin 2(t-4+4)\}$

$= e^{-4s}\mathcal{L}\{\sin 2(t+4)\}$

$= e^{-4s}\mathcal{L}\{\sin 2t\cos 8 + \cos 2t \sin 8\}$

$= e^{-4s}\left(\cos 8\cdot\dfrac{2}{s^2+4} + \sin 8\,\dfrac{s}{s^2+4}\right)$

Example. $\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\}$

use $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}=t$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s^2}\right\}$$

$$= t - u_2(t)(t-2)$$

$$= \begin{cases} t & t<2 \\ t-(t-2)=2, & t>2 \end{cases}$$

Solve the IVP

$$y'+y = 1-u_1(t) \qquad y(0)=0$$

$$sY(s)+Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$(s+1)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$Y(s) = \frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)}$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{e^{-s}}{s} + \frac{e^{-s}}{s+1}$$

$\mathcal{L}^{-1}$

$$y(t) = 1 - e^{-t} - u_1(t) + \underbrace{u_1(t)\,e^{-(t-1)}}_{\text{see after}}$$

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s+1}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \text{ delayed by } 1$$
$$= e^{-t} \text{ delayed by } 1$$
$$= u_1(t)\, e^{-(t-1)}$$

$$y(t) = \begin{cases} 1 - e^{-t} & t<1 \\ -e^{-t} + e^{1-t} & t>1 \end{cases}$$

Solve the IVP

$$y'' + y = g(t)$$
$$y(0) = 0, \quad y'(0) = 0$$

$$g(t) = \begin{cases} e^{-t} & t<2 \\ 0 & t>2 \end{cases}$$

$$s^2 Y(s) + Y(s) = \mathcal{L}\{g(t)\}$$
$$= \mathcal{L}\{e^{-t}(1 - u_2(t))\}$$
$$= \mathcal{L}\{e^{-t}\} - \mathcal{L}\{e^{-t}u_2(t)\}$$
$$= \frac{1}{s+1} - e^{-2s-2}\cdot\frac{1}{s+1}$$

$$\mathcal{L}\{u_2(t)e^{-t}\} = \mathcal{L}\{u_2(t)\,e^{-(t-2+2)}\}$$
$$= e^{-2s}\mathcal{L}\{e^{-(t+2)}\}$$
$$= e^{-2s}\mathcal{L}\{e^{-2}\cdot e^{-t}\}$$
$$= e^{-2s-2}\mathcal{L}\{e^{-t}\}$$
$$= e^{-2s-2}\cdot\frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s^2+1)(s+1)} - e^{-2}e^{-2s}\cdot\frac{1}{(s^2+1)(s+1)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = h(t) - e^{-2}u_2(t)\,h(t-2)$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\}$$

$$\frac{1}{(s^2+1)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)(s+1)$$
$$= As^2 + A + Bs^2 + (C+B)s + C$$

$$A+B=0,\quad C+B=0,\quad A+C=1$$

$$A=\frac{1}{2},\quad B=-\frac{1}{2}\quad C=-\frac{1}{2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}\cdot\cos t + \frac{1}{2}\sin t$$

delayed by 2

$$u_2(t)\left[\frac{1}{2}e^{-(t-2)} - \frac{1}{2}\cos(t-2) + \frac{1}{2}\sin(t-2)\right]$$

$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right) - e^{-2}u_2(t)\left[\frac{1}{2}e^{-t+2} - \frac{1}{2}\cos(t-2) + \frac{1}{2}\sin(t-2)\right]$$

# Appendix

Write $f(t) = \begin{cases} 6 & 0 \le t < 1 \\ e^t & 1 < t < 2 \\ \frac{t-1}{2} & 2 < t < 3 \\ 4 & t > 3 \end{cases}$

$$f(t) = 6 + \underbrace{u_1(t)\left(e^t - 6\right)}$$

– the fcn changes from 6 to $e^t$ at $t = 1$

$$+ \underbrace{u_2(t)\left(\frac{t-1}{2} - e^t\right)}$$

the fcn changes from $e^t$ to $\frac{t-1}{2}$ at $t=2$

$$+ \underbrace{u_3(t)\left(4 - \frac{t-1}{2}\right)}$$

the fcn changes from $\frac{t-1}{2}$ to 4 at $t = 3$