Lecture 24 (March 11)

$e^{at}y(t)$ $\quad$ $Y(s-a)$ $\qquad$ $Y(s) = \mathcal{L}\{y(t)\}$

$u_c(t)y(t-c)$ $\quad$ $e^{-cs}Y(s)$

① $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\}$

$$Y(s-4) = \frac{1}{(s-4)^2}, \quad Y(s) = \frac{1}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}\cdot y(t) = e^{4t}\cdot t$$

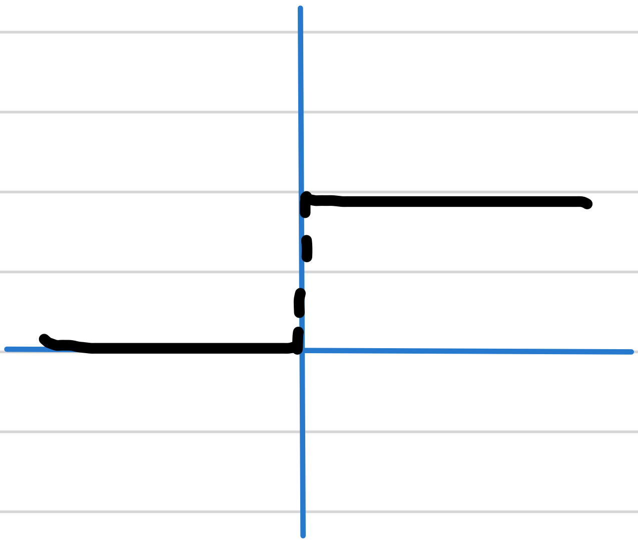
② . $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$

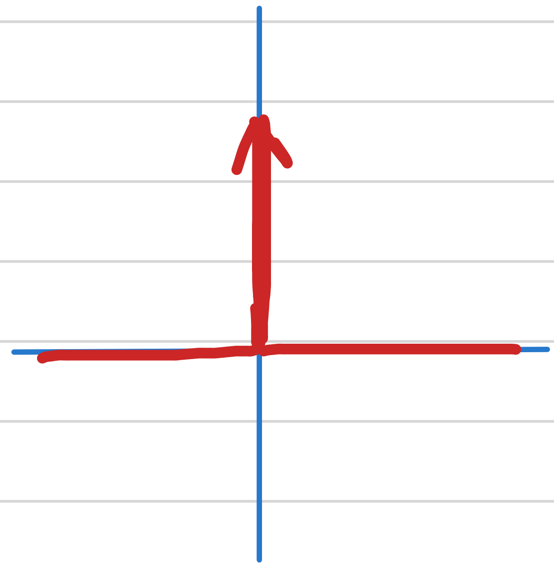
$$\frac{e^{-2s}}{s^2} = e^{-2s}\cdot \underbrace{\frac{1}{s^2}}_{Y(s)} = e^{-2s}Y(s)$$

$$y = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}^{-1}\{e^{-2s}Y(s)\} = u_2(t)\,y(t-2) = u_2(t)(t-2)$$

Dirac delta function $\delta(t)$





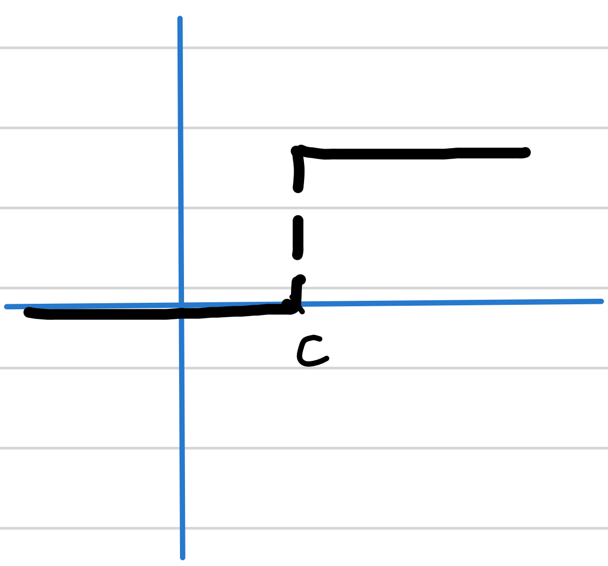
$u_0(t)$

$\delta(t) = \frac{d}{dt} u_0(t)$

not satisfactory characterization of what happens at $t=0$

$\delta(t) = 0 \,, \quad t \neq 0$

"$\delta(t) = +\infty$" at $t=0$



$u_c(t)$

$\delta(t-c)$

$$\int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt$$ $f(t)$ smooth

$$= \int_{-A}^{A} f(t)\,\delta(t)\,dt$$ any $A>0$, $\delta(t)=0$ outside $[-A,A]$

$$= \int_{-A}^{A} f(t)\,u_0'(t)\,dt$$

integration by parts

$$= f(t)\,u_0(t)\Big|_{-A}^{A} - \int_{-A}^{A} u_0(t) f'(t)\,dt$$

$$= f(t)\,u_0(t)\Big|_{-A}^{A} - \int_{0}^{A} f'(t)\,dt$$

$$= f(A)\underbrace{u_0(A)}_{=1} - f(-A)\underbrace{u_0(-A)}_{=0} - [f(A) - f(0)]$$

$$= f(A) - f(A) + f(0)$$

$$= f(0).$$

Similarly
$$\int_{-\infty}^{+\infty} f(t)\,\delta(t-c)\,dt$$
$$= f(c)$$

$$\int_{-\infty}^{+\infty} \delta(t-c)\,dt = \int_{-\infty}^{+\infty} 1\cdot\delta(t-c)\,dt$$
$$= \underline{1}.$$

$$\mathcal{L}\{\delta(t-c)\} = \int_0^{+\infty} e^{-st}\,\delta(t-c)\,dt \qquad c \geq 0$$

$$= \int_{-\infty}^{+\infty} e^{-st}\,\delta(t-c)\,dt$$ integral contributed from $(-\infty, 0)$ is 0

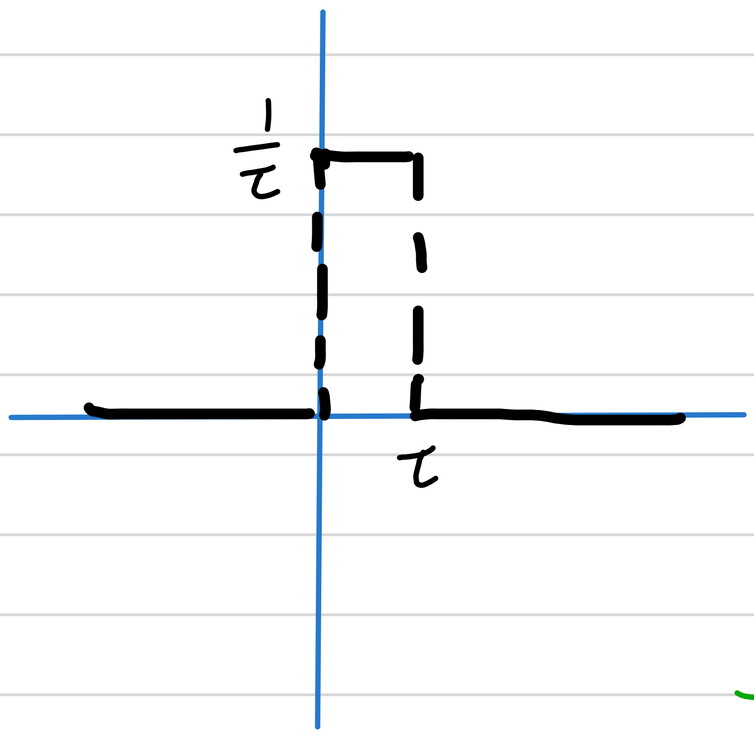
$$= e^{-sc}$$

$\mathcal{L}\{\delta(t)\} = 1$

Another definition

A constant force over a short period of time, $f(t) = \begin{cases} F & a \leq t \leq b \\ 0 & \text{elsewhere} \end{cases}$.

$\int_a^b f(t)\,dt = F(b-a)$: total impulse over $[a,b]$



$$d_\tau(t) = \begin{cases} \frac{1}{\tau} & 0 \leq t < \tau \\ 0 & t < 0 \text{ or } t \geq \tau \end{cases}$$

$$\int_{-\infty}^{+\infty} d_\tau(t)\,dt = \int_0^{\tau} \frac{1}{\tau}\,dt = 1$$

$\tau \to 0, \quad d_\tau(t) \to \delta(t)$

$$d_\tau(t) = \frac{1}{\tau}\left(1 - u_\tau(t)\right) \qquad t \geq 0$$

$$\mathcal{L}\{d_\tau(t)\} = \frac{1}{\tau}\mathcal{L}\{1-u_\tau(t)\}$$

$$= \frac{1}{\tau}\left[\frac{1}{s} - \frac{e^{-\tau s}}{s}\right]$$

$$= \frac{1-e^{-\tau s}}{\tau s}$$

$$\xrightarrow{\tau \to 0} 1$$

$$\lim_{x\to 0} \frac{1-e^{-x}}{x} = \lim_{x \to 0} \frac{e^{-x}}{1} = 1$$

$$d_\tau(t) \longrightarrow \delta(t).$$

$\delta(t)$: provides a unit impulse at a single instant

Just think $\delta(t)$ to be the unique "function" whose Laplace Transform is 1

$\delta(t-c)$ ... e^{-cs}

Solve the IVP.

$$2y'' + y' + 2y = \delta(t-5) \qquad y(0),\ y'(0) = 0$$

$$2s^2 Y(s) + sY(s) + 2Y(s) = e^{-5s}$$

$$Y(s) = \frac{e^{-5s}}{2s^2+s+2} = \frac{1}{2} e^{-5s} \frac{1}{s^2+\frac{s}{2}+1}$$

$$= \frac{1}{2} e^{-5s} \frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}$$

$$y(t) = \frac{1}{2} u_5(t) \mathcal{L}^{-1}\left\{ \frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}} \right\}\Big|_{t \to t-5}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}\right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+\frac{15}{16}}\Big|_{s \to s+\frac{1}{4}} \right\}$$

$$= \frac{4}{\sqrt{15}} e^{-\frac{t}{4}} \mathcal{L}^{-1}\left\{ \frac{\frac{\sqrt{15}}{4}}{s^2+\frac{15}{16}} \right\}$$

$$= \frac{4}{\sqrt{15}} e^{-\frac{t}{4}} \cdot \sin \frac{\sqrt{15}}{4} t$$

$$y(t) = \frac{1}{2} u_5(t) \cdot \frac{4}{\sqrt{15}} e^{-\frac{t-5}{4}} \sin \frac{\sqrt{15}}{4}(t-5)$$

$$= \begin{cases} 0 & 0 \le t < 5 \\ \frac{2}{\sqrt{15}} e^{-\frac{t-5}{4}} \sin \frac{\sqrt{15}}{4}(t-5) & t \ge 5 \end{cases}$$

IVP for spring system

$$y'' + y = A\,\delta(t - \tfrac{\pi}{2}), \quad y(0) = 1, \quad y'(0) = 0$$

$$s^2 Y(s) - s + Y(s) = A e^{-\frac{\pi}{2}s}$$

$$Y(s) = \frac{s}{s^2+1} + A \cdot \frac{e^{-\frac{\pi}{2}s}}{s^2+1}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = \cos t + A u_{\frac{\pi}{2}}(t) \sin(t - \tfrac{\pi}{2})$$

$$= \begin{cases} \cos t & 0 \le t < \pi/2 \\ (1-A)\cos t & t \ge \frac{\pi}{2} \end{cases}$$