Lecture 25 (March 13)

Convolution.

$$\mathcal{L}^{-1}\{F(s)\,G(s)\}$$

$\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}\{g(t)\} = G(s)$

$$\cancel{=}\ f(t)\,g(t)$$

Theorem. $\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau$

$$= \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$\overset{\text{definition}}{=} (f*g)(t)$$

"convolution of f and g"

$$f*g = g*f$$

$$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$u = t-\tau$

$$= -\int_t^0 f(u)\,g(t-u)\,du$$

$$= \int_0^t g(t-u)f(u)\,du$$

$\tau \in (0,t)$, $u \in (t,0)$

$d\tau = -du$

$$\mathcal{L}\{f*g\} = F(s)\cdot G(s)$$

Example. $t^2 * t$

$$t^2 * t = \int_0^t \tau^2 (t-\tau)\,d\tau$$
$$= t\frac{\tau^3}{3}\Big|_0^t - \frac{\tau^4}{4}\Big|_0^t$$
$$= \frac{t^4}{3} - \frac{t^4}{4}$$
$$= \frac{t^4}{12}$$

OR. $\mathcal{L}\{t^2 * t\} = \mathcal{L}\{t^2\} \cdot \mathcal{L}\{t\}$
$$= \frac{2}{s^3} \cdot \frac{1}{s^2} = \frac{2}{s^5}$$

$$t^2 * t = \mathcal{L}^{-1}\left\{\frac{2}{s^5}\right\} = \frac{1}{12}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{12}t^4$$

Example: $(\cos t) * 1$

$$\mathcal{L}\{(\cos t) * 1\} = \mathcal{L}\{\cos t\}\,\mathcal{L}\{1\}$$
$$= \frac{s}{s^2+1} \cdot \frac{1}{s}$$
$$= \frac{1}{s^2+1}$$
$$(\cos t) * 1 = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$(f * \delta)(t) \cdot \xrightarrow{\mathcal{L}} F(s) \cdot 1 = F(s)$$

$$f(t) \xleftarrow{\mathcal{L}^{-1}}$$

$$(f * \delta)(t) = f(t)$$

Solutions of IVPs can be written in terms of convolutions

$$ay'' + by' + cy = g(t) \qquad y(0)=0, \; y'(0)=0$$

$$(as^2+bs+c)Y(s) = G(s)$$

g: input

y: response

$$Y(s) = \frac{1}{as^2+bs+c} \cdot G(s)$$

$$= H(s)G(s)$$

$H(s) = \frac{1}{as^2+bs+c}$: transfer function

Let $h(t) = \mathcal{L}^{-1}\{H(s)\}$

$$y(t) = (h * g)(t)$$

What is $h(t)$?

$$(as^2+bs+c)H(s) = 1$$

$$\uparrow \mathcal{L}$$

$$ah''(t) + bh'(t) + ch(t) = \delta(t), \quad h(0)=0, \; h'(0)=0$$

$h(t)$ is the response to unit impulse at $\{t=0\}$ (called the impulse response) y is the convolution of the impulse response and the input function. $g(t)$

Example. $y'' + 9y = g(t)$, $y(0) = -1$, $y'(0) = 3$

$$s^2 Y(s) - s y(0) - y'(0) + 9Y(s) = G(s)$$

$$(s^2+9)Y(s) + s - 3 = G(s)$$

$$Y(s) = \frac{1}{s^2+9} G(s) - \frac{s}{s^2+9} + \frac{3}{s^2+9}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin 3t$

$$y(t) = \frac{1}{3}\sin 3t * g(t) - \cos 3t + \sin 3t$$

$$= \frac{1}{3}\int_0^t \sin 3(t-\tau) g(\tau)\, d\tau - \cos 3t + \sin 3t.$$

Now, let's prove

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

$$F(s)\,G(s) = \int_0^{+\infty} e^{-su} f(u)\,du \int_0^{+\infty} e^{-sv} g(v)\,dv$$

$$= \int_0^{+\infty} \int_0^{+\infty} e^{-s(u+v)} f(u) g(v)\,du\,dv$$

change of variable

fix v, $u+v=t$

$u = t-v$

$$= \int_0^{+\infty} g(v) \int_v^{+\infty} e^{-st} f(t-v)\,dt\,dv$$

$u \in (0, \infty)$

$t \in (v, \infty)$

$du = dt$

$$= \int_0^{+\infty} e^{-st} \int_0^{t} g(v) f(t-v)\,dv\,dt$$

$$= \int_0^{+\infty} e^{-st} (f*g)(t)\,dt$$

$$= \mathcal{L}\{f*g\}.$$