Lecture 3 (Jan 11)

First order differential equation.

$$\frac{dy}{dt} = f(t,y)$$

f : a known function, examples: $f(t,y) = 0$

$f(t,y) = y - t^2$

$f(t,y) = -\frac{t}{y}$

3 Topics:

① Direction Fields: Graphical Representation of Differential Equations

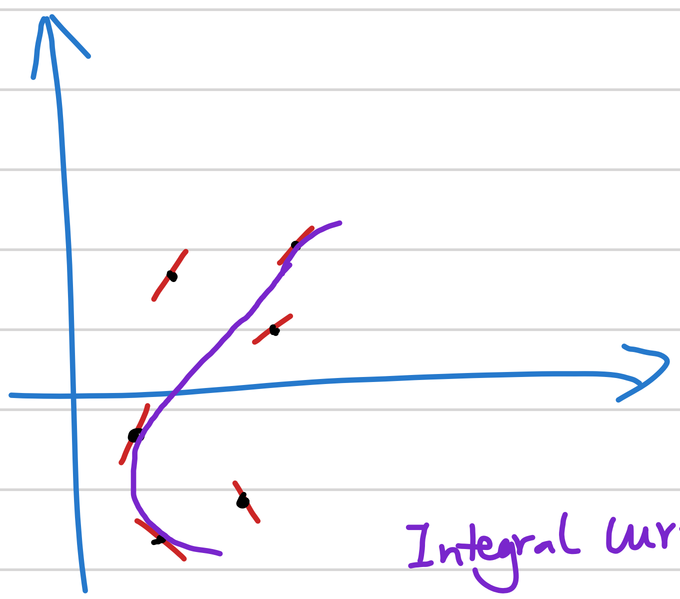
② Find formulas of solutions linear / nonlinear

③ Euler's method: a computer uses to solve a DE (approximately)

Geometrical interpretation of $\frac{dy}{dt} = f(t,y)$:

- a solution $y(t)$ passing (t_0, y_0) ($y(t_0) = y_0$), the slope of $y(t)$ at $(t_0, y(t_0))$ is just $f(t_0, y_0)$

Direction Field.



at each point (t,y) on the plane, evaluate $f(t,y)$, draw a line segment with slope $= f(t,y)$

Integral curve: tangent to the line elements at all points along the curve.

$y(t)$ solution of the DE $\Longleftrightarrow$ graph of $y(t)$ is an integral curve

$y'(t) = f(t,y(t))$ $\Longleftrightarrow$ slope of $y(t)$ = slope of direction field (at $(t,y(t))$)

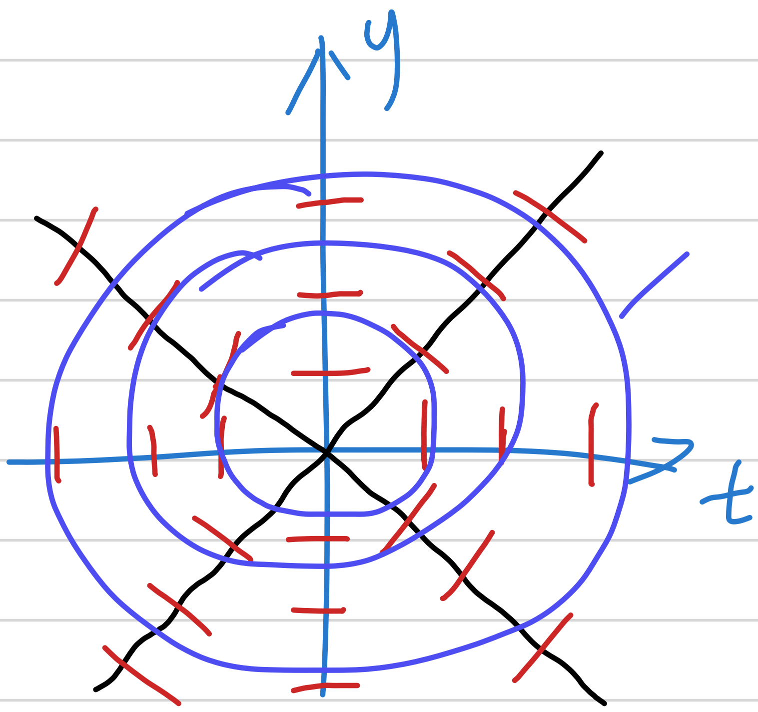
$$y'(t) = f(t,y(t))$$

Initial Value Problem $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$ (t_0, y_0) given

(Existence and Uniqueness) There is exactly one solution to IVP.

- There are conditions: f is a differentiable function
- We don't know how far a solution can go, "a unique solution defined on a neighborhood of t_0.

Example: $\frac{dy}{dt} = -\frac{t}{y}$ (non linear)



integral curves (circles)

① All the lines at $y=0$ have slope ∞

② All the lines at $t=0$ have slope 0

③ All the lines at $y=t$ have slope -1

④ All the lines at $y=-t$ have slope 1

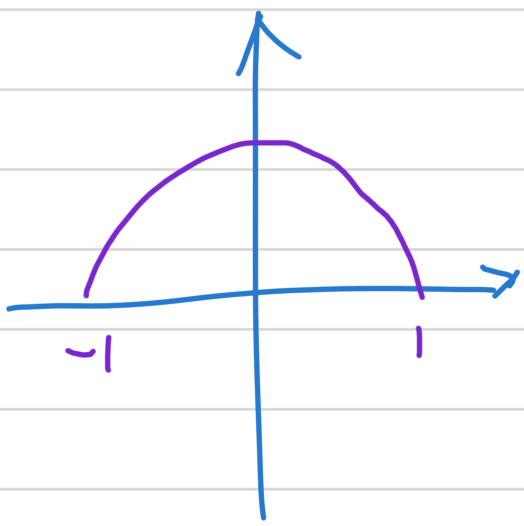
at $y=ct$, the lines have slope $-\frac{1}{c}$
perpendicular to $\{y=ct\}$

Integral curves : $t^2+y^2=C$.

$$y = \pm\sqrt{C-t^2}$$

Initial value problem

$$\begin{cases} y' = -\frac{t}{y} \\ y(0) = 1 \end{cases}$$



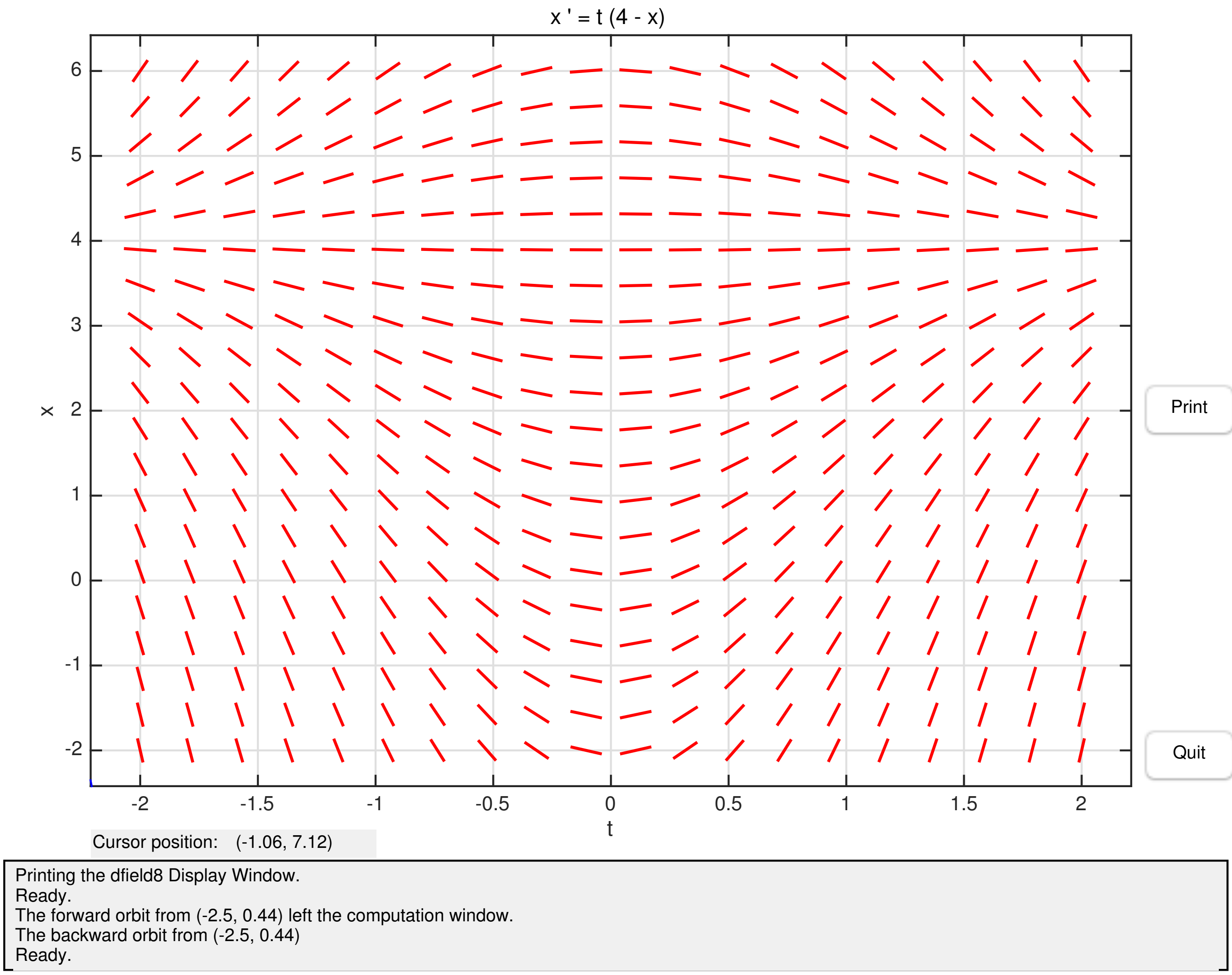
$1 = \sqrt{C-0} \Rightarrow C=1$

$y = \sqrt{1-t^2}$ only defined on $[-1, 1]$.

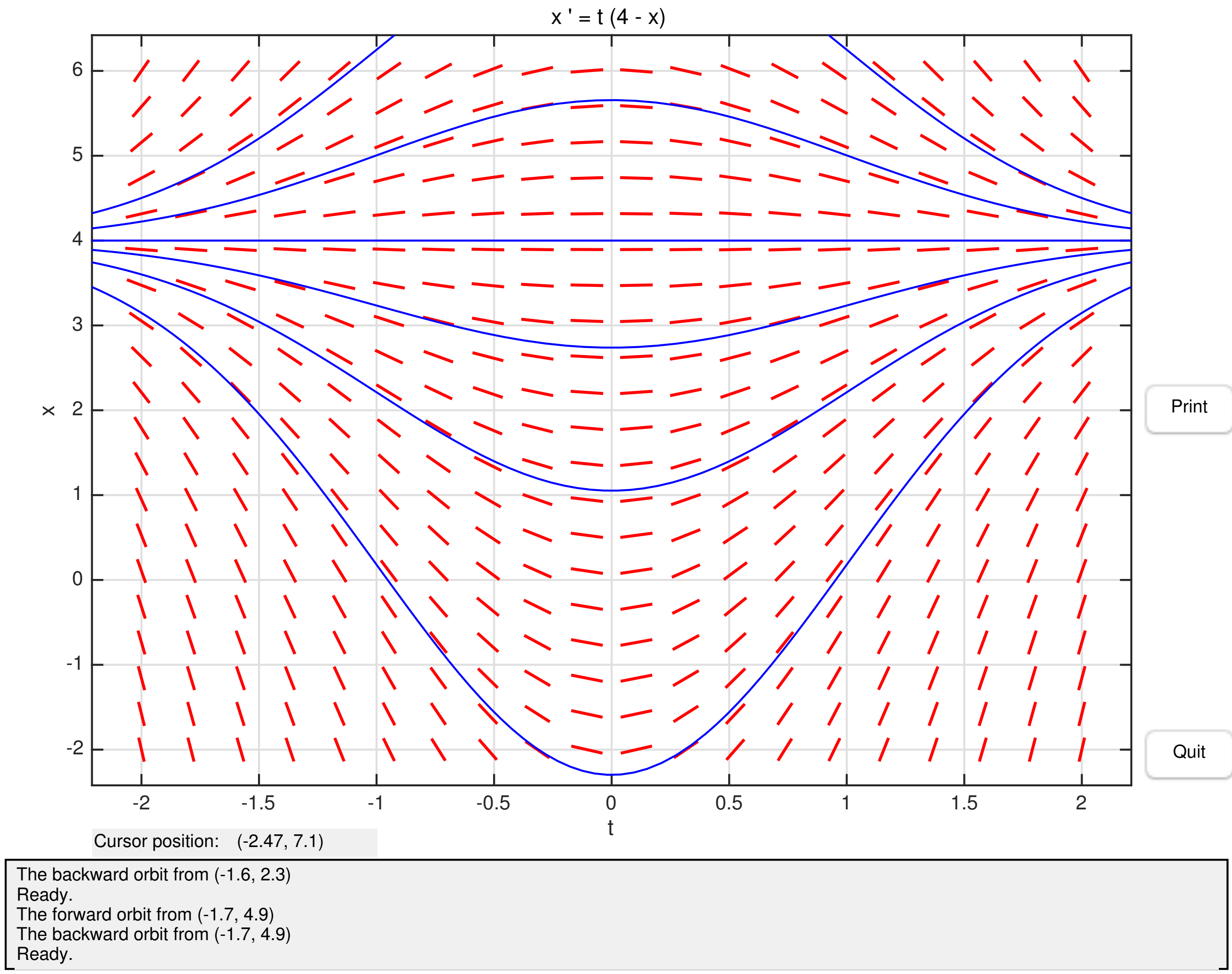
$y = \sqrt{1-t^2}$
$y = -\sqrt{1-t^2}$ They both solve $\begin{cases} y' = -\frac{t}{y} \\ y(-1) = 0 \end{cases}$

Contradictory to existence and uniqueness theorem?

No, $-\frac{t}{y}$ not differentiable at $y=0$



- lines at $y=4$ have slope 0
- lines at $t=0$ have slope 0



Integral curves shown above

$y \equiv 4$ equilibrium sol'n (stable)

as $t \to \infty$ $y(t) \to 4$

solutions start below 4 stay below 4
above 4 stay above 4

Solve the equation (analytically)

$$\frac{dy}{dt} = t(4-y)$$

$$\frac{1}{y-4}\frac{dy}{dt} = -t$$

$$\frac{d}{dt}\ln|y-4| = t$$

$$\ln|y-4| = -\frac{t^2}{2} + C$$

$$|y-4| = e^C e^{-\frac{t^2}{2}}$$

$$y = 4 + Ce^{-\frac{t^2}{2}}$$

$t \to \infty,\quad y \to 4$