Lecture 4 (Jan 14)

A newly constructed fish pond contains 3000 liters of water. Unfortunately the pond has been contaminated with 5kg of a toxic chemical. The pond's filtering system removes water from the pond at a rate of 200 liters/min, removes 40% of the chemical, and return the same volume of water to the pond.

Differential Equation for the total mass of the chemical
(t measured in minutes)

$m(t)$: total mass $\qquad m(0)=5$

$$\frac{dm}{dt} = -\text{rate of chemical being removed}$$

concentration of chemilal : $\frac{m(t)}{3000 \text{ Liters}}$

rate of chemilal being removed
= rate of water through the filter × concentration × 40%

$$= \frac{m(t)}{3000 \text{ liters}} \times \frac{200 \text{ liters}}{\text{min}} \times 0.4 = \frac{2m(t)}{75 \text{ min}}$$

$$\frac{dm}{dt} = -\frac{2}{75} m$$

DE for concentration (in kg/Liter) of the chemilal in the pond

$$c = \frac{m}{3000}$$

$$\frac{dc}{dt} = \frac{1}{3000}\frac{dm}{dt} = \frac{1}{3000} \times \left(-\frac{2}{75} m\right) = -\frac{2}{75} c$$

$$\frac{dy}{dt} = y(y-1)(y-3)$$

Find the equilibrium solutions

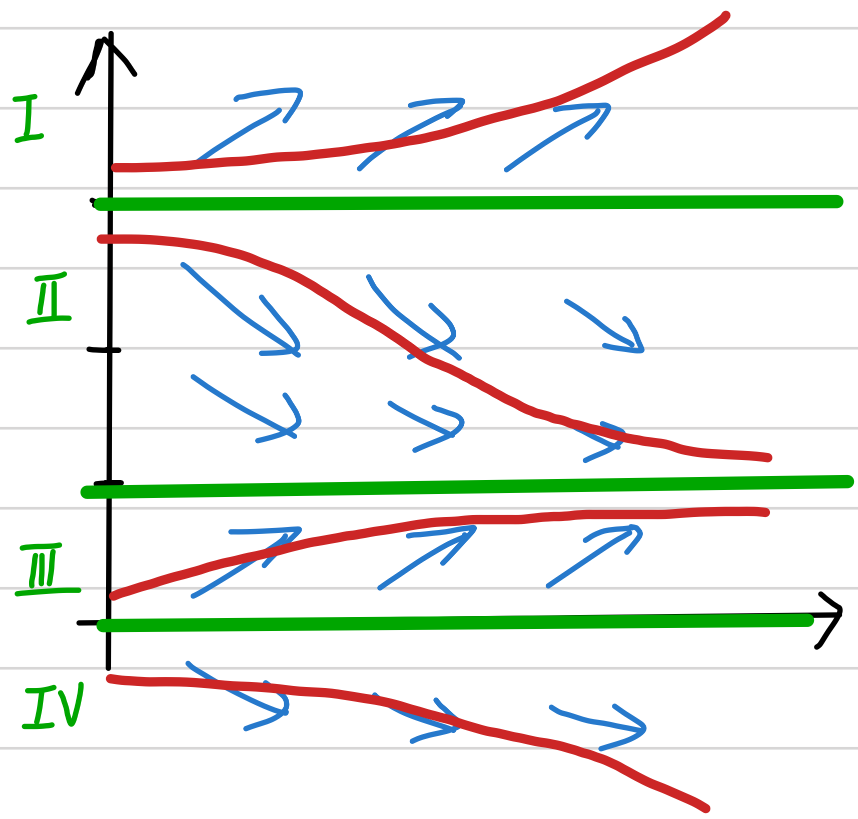
A solution y is called an equilibrium solution if $y(t) = \text{constant}$

So $\frac{dy}{dt} = 0$, and

$$y(y-1)(y-3) = 0$$

Equilibrium solutions:

$y=0$; $y=1$; $y=3$



If $y<0$, $y(y-1)(y-3) < 0$

If $0<y<1$, $y(y-1)(y-3) > 0$

If $1<y<3$, $y(y-1)(y-3) < 0$

If $y>3$ $y(y-1)(y-3) > 0$

solution starts in one region stays in that region (by Existence and unique theorem)

$y=0$: unstable equilibrium, solutions start near 0 move away

$y=1$: stable equilibrium, solutions start near 1 move towards it

$y=3$: unstable equilibrium, solutions start near 3 move away

First order differential equation

$$\frac{dy}{dt} = f(t,y)$$

Examples: $f(t,y) = t - y^2$, $f(t,y) = t^2 - y$

If $f(t,y) = F(t)G(y)$, then the DE is "separable"

Examples: $f(t,y) = ty$, $f(t,y) = \frac{\sin t}{y^2}$. $f(t,y) = e^{5t+3y}$

$$\frac{dy}{dt} = F(t)G(y)$$

$$\frac{dy}{G(y)} = F(t)dt$$

Integrate both sides

$$\int \frac{dy}{G(y)} = \int F(t)dt + C$$

For An Initial Value Problem : find C using initial condition

Try to solve $y(t)$

Example: $\frac{dy}{dt} = -\frac{t}{y}$

Notice $-\frac{t}{y} = (-t) \times \frac{1}{y}$

$y\,dy = -t\,dt \qquad y(0)=1$

$\int y\,dy = \int -t\,dt$

$\frac{1}{2}y^2 = -\frac{t^2}{2} + C$

$t^2 + y^2 = C$

$0^2 + 1 = C \Rightarrow C = 1$

Implicit solution

$t^2 + y^2 = 1$

Explicit solution

$y = \pm\sqrt{1-t^2}$

$y(0)=1 \Rightarrow y = \sqrt{1-t^2}$

Example: $\frac{dy}{dt} = \frac{3t^2+4t+2}{2(y-1)} \qquad y(0) = -1$

Rewrite $2(y-1)\,dy = (3t^2 + 4t + 2)\,dt$

$y^2 - 2y = t^3 + 2t^2 + 2t + C$

Use IC $\quad 1 + 2 = C$

$\Rightarrow C = 3$

$y^2 - 2y = t^3 + 2t^2 + 2t + 3$ (quadratic eq. for y)

you know how to solve it if you know how to solve $x^2 - 2x = b$

$y = 1 \pm \sqrt{t^3 + 2t^2 + 2t + 4}$

Use IC again

$y = 1 - \sqrt{t^3 + 2t^2 + 2t + 4}$

Example: $\frac{dy}{dt} = \frac{4t - t^3}{4 + y^3}$ $\quad y(0) = 1$

$(4 + y^3)\,dy = (4t - t^3)\,dt$

$4y + \frac{1}{4}y^4 = 2t^2 - \frac{1}{4}t^4 + C$

$y^4 + t^4 + 16y - 8t^2 = C$

$y(0) = 1 \Rightarrow 1 + 16 = C$

$\Rightarrow C = 17$

$y^4 + t^4 + 16y - 8t^2 = 17$ (implicit solution)

Not easy to write the explicit solution