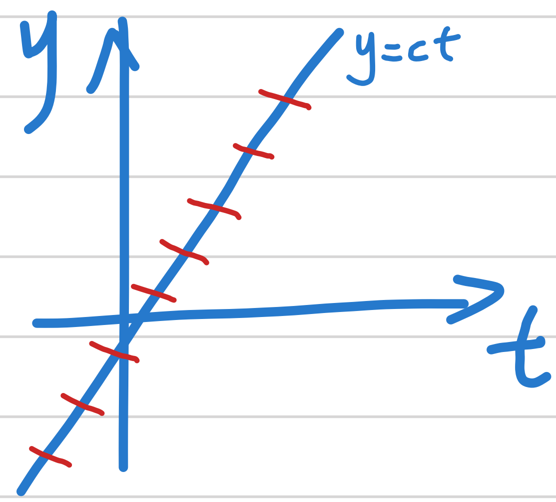
Lecture 5 (Jan. 16)

$\frac{dy}{dt} = f\left(\frac{y}{t}\right)$ homogeneous equation

Not separable in general.

Special property of direction: on a line $\{y=ct\}$, same slope $f(c)$



Example: $\frac{dy}{dt} = \frac{t^2+3y^2}{2ty}$

$$\frac{t^2+3y^2}{2ty} = \frac{1+3\left(\frac{y}{t}\right)^2}{2\,\frac{y}{t}}$$

Substitution $v(t) = \frac{y(t)}{t}$, $y(t) = t\,v(t)$

$$\frac{dy}{dt} = v(t) + t\frac{dv}{dt} = \frac{1+3v^2}{2v}$$

$$t\frac{dv}{dt} = \frac{1+3v^2}{2v} - v = \frac{1+v^2}{2v}$$

$$\frac{dv}{dt} = \frac{1+v^2}{2vt} \quad \text{(separable)}$$

$$\frac{2v\,dv}{1+v^2} = \frac{dt}{t}$$

$$\ln(1+v^2) = \ln|t| + C$$

$$1+v^2 = e^C|t| = \pm e^C t = ct$$

Remember $v = \frac{y}{t}$

$$1 + \left(\frac{y}{t}\right)^2 = ct$$

$$t^2 + y^2 = ct^3$$

First order linear differential equation:

$$a(t)y'(t) + b(t)y(t) = C(t)$$

Examples: $y'(t) + t\,y(t) = \cos t$

$$y'(t) - 5y(t) = e^{-3t}$$

$$y'(t) - (\cos t)y = \sin t$$

Standard form:

$$y'(t) + p(t)y(t) = g(t)$$ (always solvable)

If $a(t) \neq 0$, you can always transform into the standard form

$$y'(t) + \frac{b(t)}{a(t)}y(t) = \frac{c(t)}{a(t)}$$

Example: $y' - 2y = 0 \qquad y(0) = 3$

Rewrite $\frac{dy}{dt} = 2y$ (separable)

$$\frac{dy}{y} = 2dt$$

$$\ln|y| = 2t + C$$

$$|y| = e^C e^{2t}$$

$$y = c\,e^{2t}$$

use $y(0) = 3$, $c = 3$

$$y(t) = 3e^{2t}$$

Example: $(4+t^2)y' + 2ty = 4e^t$

An observation: $(4+t^2)y' + 2ty = [(4+t^2)y]'$

$$[(4+t^2)y]' = 4e^t$$

$$\int [(4+t^2)y]'\,dt = \int 4e^t\,dt$$

Use the Fundamental Theorem of Calculus

$$(4+t^2)y = 4e^t + C$$

$$y = \frac{4e^t}{t^2+4} + \frac{C}{t^2+4}$$

Example: $y' - 2y = 5e^{4t}, \quad y(0) = 3$

Not separable.

$\boxed{\text{A trick:}}$ Multiply both sides by $\boxed{e^{-2t}}$ — integrating factor

$$e^{-2t}y'(t) - 2e^{-2t}y(t) = 5e^{2t}$$

Observation: $e^{-2t}y'(t) - 2e^{-2t}y(t) = \left(e^{-2t}y(t)\right)'$

$$\left(e^{-2t}y(t)\right)' = 5e^{2t}$$

Integrate both sides

$$e^{-2t}y(t) = \frac{5}{2}e^{2t} + C$$

Divide both sides by e^{-2t}

$$y(t) = \frac{5}{2}e^{4t} + Ce^{2t}$$

Use I.C.

$$3 = \frac{5}{2} + C \Rightarrow C = \frac{1}{2}$$

$$y(t) = \frac{5}{2}e^{4t} + \frac{1}{2}e^{2t}$$

How to choose the integrating factor. $\mu(t)$?

$$\mu(t)\,y'(t) - 2\mu(t)\,y(t) \overset{\Uparrow}{=} (\mu(t)\,y(t))'$$

choose one proper $\mu(t)$

$$= \mu(t)\,y'(t) + \mu'(t)\,y(t)$$

what we need?

$$-2\mu(t) = \mu'(t)$$

$$\frac{d\mu}{dt} = -2\mu \quad (\text{A D.E. for } \mu(t))$$

$$\frac{d\mu}{\mu} = -2\,dt$$

$$\ln|\mu(t)| = -2t + C$$

$$\mu(t) = c\,e^{-2t}$$

Any c would work, why not pick the simplest one $c=1$

Now let us consider a general form

$$y' + ay = g(t) \qquad a \text{ is a constant.}$$

$$\mu(t)\, y' + a\mu(t)\, y = \mu(t)\, g(t)$$

We want $\quad \mu(t)\, y' + a\,\mu(t)\, y = (\mu(t)\, y)' = \mu(t)\, y' + \mu'(t)\, y$

choose $\mu(t)$, such that

$$\mu'(t) = a\mu(t)$$

pick $\mu(t) = e^{at}$

$$(e^{at} y(t))' = e^{at} g(t)$$

$$e^{at} y(t) = \int e^{at} g(t)\, dt + C$$

$$e^{at} y(t) = \int_{t_0}^{t} e^{as} g(s)\, ds + c$$

$$y(t) = e^{-at} \int_{t_0}^{t} e^{as} g(s)\, ds + c\, e^{-at}$$

Newton's law of cooling.

$$\frac{dT}{dt} = k(T_A - T)$$

$$\frac{dT}{dt} + kT = kT_A(t)$$

T: temperature of the object

T_A: ambient temp

k: constant of proportionality

Use the formula. $(k>0)$

$$T(t)= e^{-kt}\int_0^t e^{ks}\,k\,T_A(s)\,ds + C e^{-kt}$$

$T(0)=T_0$, $C=T_0$

$$T(t)= \underbrace{e^{-kt}\int_0^t e^{ks}\,k\,T_A(s)\,ds}_{\text{steady state solution}} + \underbrace{T_0 e^{-kt}}_{\text{goes to 0 as } t\to\infty}$$