Lecture 6 (Jan 18)

$$y' + p(t)y = g(t)$$

$$\mu(t)\, y'(t) + \mu(t)\, p(t)\, y(t) = \mu(t)\, g(t)$$

$$(\mu(t)\, y(t))' = \mu(t)\, y'(t) + \mu'(t)\, y(t)$$

We choose $\mu(t)$ such that

$$\mu'(t) = \mu(t)\, p(t) \qquad \text{(separable)}$$

The eq. is transformed to

$$[\mu(t)\, y(t)]' = \mu(t)\, g(t)$$

$$\frac{d\mu}{\mu} = p(t)\, dt$$

Just assume $\mu(t) > 0$

$$\ln \mu(t) = \int p(t)\, dt$$

$$\mu(t) = e^{\int p(t)\, dt}$$

$$\mu(t)\, y(t) = \int \mu(t)\, g(t)\, dt + C$$

$$y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\, g(s)\, ds + \frac{C}{\mu(t)}$$

Example: $y' + \frac{2}{t} y = t$, $y(2) = 2$

Find Integrating factor

$$\frac{d\mu}{dt} = \frac{2}{t}\mu$$

$$\frac{d\mu}{\mu} = \frac{2}{t}\, dt$$

$$\ln \mu(t) = 2\ln|t| + C = \ln t^2 + C$$

$$\mu(t) = t^2$$

Multiply both sides by t^2

$$t^2 y' + 2ty = t^3$$

$$(t^2 y)' = t^3$$

Integrate

$$t^2 y = \int t^3 \, dt$$

$$= \frac{1}{4} t^4 + C$$

$$y = \frac{1}{4} t^2 + \frac{C}{t^2}$$

$y(2) = 2$, $C = 4$. $\quad y(t) = \frac{t^2}{4} + \frac{4}{t^2}$

Example: $\quad y' - 2ty = t$

$$y(0) = 1$$

Integrating factor $\mu(t)$ satisfies

$$\frac{d\mu}{dt} = (-2t)\mu$$

$$\frac{d\mu}{\mu} = -2t\,dt$$

$(\mu > 0)$ $\quad \ln \mu(t) = -t^2$

$\mu(t) = e^{-t^2}$

$(e^{-t^2} y)' = t e^{-t^2}$

$e^{-t^2} y = \int t e^{-t^2} dt$

$= -\frac{1}{2} e^{-t^2} + C$

$y = -\frac{1}{2} + C e^{-t^2}$

$y(0) = 1 \Rightarrow \quad C = \frac{3}{2}$

$y(t) = -\frac{1}{2} + \frac{3}{2} e^{-t^2}$

Example: $y' - 2ty = 3$

$y(0) = 1$

$(e^{-t^2} y)' = 3 e^{-t^2}$

$e^{-t^2} y = \underline{3 \int e^{-t^2} dt}$ (can't find antiderivative)

$e^{-t^2} y = 3 \int_0^t e^{-s^2} ds + C$

$y(0) = 1 \Rightarrow C = 1$

$y(t) = 3 e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2}$

Another way of solving first order linear differential equation:

Define L: $Ly := y'(t) + p(t)y(t)$

DE: $Ly = g$ $g=0$: homogeneous equation

$g \neq 0$: inhomogeneous equation

Important properties of L (y_1, y_2 two functions)

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$

$$L(Cy_1) = CL(y_1)$$

Let's verify these

$$\begin{aligned} L(y_1+y_2) &= (y_1+y_2)' + p(t)(y_1+y_2) \\ &= y_1' + y_2' + p(t)y_1 + p(t)y_2 \\ &= (y_1' + p(t)y_1) + (y_2' + p(t)y_2) \\ &= L(y_1) + L(y_2) \end{aligned}$$

$$\begin{aligned} L(Cy_1) &= (Cy_1)' + p(t)Cy_1 \\ &= Cy_1' + Cp(t)y_1 \\ &= C(y_1' + p(t)y_1) \\ &= C\,L(y_1) \end{aligned}$$

$Ly = y' + p(t)y = g(t)$, Find the general solutions

Divide the the task into two part

Part 1: $Ly_s = y_s' + p(t)y_s = g(t)$

Find a solution (good guess necessary)

Part 2: $Ly_g = y_g' + p(t)y_g = 0$

Find the general solutions for this homogeneous equation.

Notice: this is a separable D.E.

Finally, $y = y_s + y_g$

$$\begin{aligned} Ly &= L(y_s + y_g) \\ &= L(y_s) + L(y_g) \\ &= g + 0 \\ &= g \end{aligned}$$

general solutions of inhomogeneous equation
= special solution of inhomogeneous equation
+ general solutions of homogeneous equation

Example $y' + 4y = 1$, $y(0) = 0$

step 1. $y' + 4y = 1$, find a special solution

$$y_s = \frac{1}{4}$$

step 2. $y' + 4y = 0$, find the general solutions

$$y_g = ce^{-4t}$$

$$y = y_s + y_g = \frac{1}{4} + ce^{-4t}$$

I.C $\Rightarrow$ $c = -\frac{1}{4}$

$$y(t) = \frac{1}{4} - \frac{1}{4}e^{-4t}$$

Example: $y' + \frac{2}{t}y = t$, $y(2) = 2$

A special solution

$$y_s = \frac{t^2}{4}$$

Find general solutions for $y' + \frac{2}{t}y = 0$

$$\frac{dy_g}{dt} = -\frac{2}{t}y_g$$

$$\frac{dy_g}{y_g} = -\frac{2}{t}dt$$

$$\ln|y_g| - 2\ln|t|$$
$$= \ln|t|^{-2}$$
$$= \ln\frac{1}{t^2}$$
$$y_g = \frac{C}{t^2}$$
$$y = \frac{t^2}{4} + \frac{C}{t^2}$$