307 Notes

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January 25, 2019

1 Modeling with Differential Equations

Example 1:
Consider a tank which is initially filled with 100L of pure water. Assume that a solution containing 50g of salt per liter enters the tank at a rate of 2L/min, and that the well-stirred mixture leaves the tank at the same rate. We wish to determine the amount of salt (in grams) in the tank at any time $t > 0$ (with time being measured in minutes).

Before we derive a differential equation, let us investigate this system. First, the amount of water in the tank is constant since water enters and leaves the tank at the same rate. Therefore, at all times the amount of water in the tank equals the amount of water which is in the tank at time $t = 0$, that is, 100L. Second, if we let $S(t)$ denote the amount of salt in the tank (that is $S(t)$ tells us how many grams of salt is in the tank) at time t, then $S(0) = 0$ since the tank initially contains only pure water, and therefore no salt.

Intuitively, we expect the amount of salt $S(t)$ to have certain properties. Since there is no salt in the tank initially, yet solution containing salt is flowing into the tank, we expect the concentration of salt to increase. Moreover, after a very long time, we expect the inflow to have "replaced" the pure water that was in the tank initially, so the concentration of salt in the tank should become close to that of the solution flowing in the tank, that is, 50g/L. After we find a formula for $S(t)$, it's important to check these observations to ensure our formula makes sense.

To determine the formula for $S(t)$, we will set up a differential equation for the quantity, and then solve it. To do so, we need to assume that $S(t)$ is differentiable (I've swept this point under the rug thus far in the course, but it's important to mention. How can we find a solution to a *differential* equation, when the solution isn't even differentiable itself!). In the set up described above, we expect the amount of salt to change gradually, so the rate of change, $\frac{dS}{dt}$ should indeed be defined for all times.

To set up the differential equation for $S(t)$, we will investigate how the amount of salt in the tank changes within short times, and then makes times arbitrarily small. Let us consider this change in between time t and a slightly larger time $t+h$ (that is, $h>0$). The difference between the amount of salt at times t and $t+h$ is given by

$$S(t+h)-S(t).$$

Another way to represent the change in the amount of salt is to look at the amount of salt that left and entered the tank. Recall that salt enters the tank at 2L/min, and each liter contains 50 grams of salt. Thus the amount of salt entering the tank in h min (in grams) is given by

$$(h\text{min})\cdot\left(2\frac{\text{L}}{\text{min}}\right)\cdot\left(50\frac{\text{g}}{\text{L}}\right)=(100h)\text{g}.$$

Now, the concentration of salt at a time t in the tank is $\frac{S(t)}{100}$g/L (recall the 100L fixed tank quantity), so since the mixture leaves the tank at a rate of 2L/min, the amount of salt leaving the tank in h min is *approximately*

$$(h\text{min})\cdot\left(\frac{S(t)}{100}\frac{\text{g}}{\text{L}}\right)\cdot\left(2\frac{\text{L}}{\text{min}}\right)=\left(\frac{S(t)}{50}h\right)\text{g}.$$

This is not the precise amount since the amount of salt will vary in between times t and $t+h$, however as previously stated, we expect only gradual changes in $S(t)$, so it should and will not matter whether we replace $S(t)$ by $S(t+h)$ or the value of S at any time between t and $t+h$. Now, the change in the amount of salt that enters the tank minus the amount that leaves the tank, is given by

$$S(t+h)-S(t)=100h-\frac{S(t)}{50}h.$$

Dividing both sides by h we obtain

$$\frac{S(t+h)-S(t)}{h}=100-\frac{S(t)}{50}.$$

Now, we make the difference between the two times arbitrarily small, that is, we let $h\to 0$. But the LHS is our difference quotient, so as $h\to 0$, we get that

$$\begin{aligned}\frac{dS}{dt}(t)&=\lim_{h\to 0}\frac{S(t+h)-S(t)}{h}\\&=\lim_{h\to 0}\left(100-\frac{S(t)}{50}\right)\\&=100-\frac{S(t)}{50}.\end{aligned}$$

We have thus derived a differential equation for the amount of salt S, indeed, since $S(0)=0$, as mentioned above, we have the initial value problem given by

$$\frac{dS}{dt}=100-\frac{S}{50},\qquad S(0)=0.$$

Solving this initial value problem is fairly straightforward as it's autonomous, so we get

$$\frac{dS}{100 - \frac{S}{50}} = dt.$$

Hence

$$\int dt = t + C$$

and

$$\int \frac{ds}{100 - \frac{S}{50}} = -50 \ln \left| 100 - \frac{S}{50} \right|.$$

Note that we can ignore this absolute value sign for the following reason: Initially, there is no salt in the tank, and the concentration of salt in the inflowing water is 50 g/L, so there cannot be more than $100 \times 50 = 5000$ grams of salt in the water. That is, $S \leq 5000$, or equivalently $100 - \frac{S}{50} \geq 0$. Hence we have the equation

$$-50 \ln \left(100 - \frac{S}{50} \right) = t + C.$$

Moving stuff around

$$\ln \left(100 - \frac{S}{50} \right) = -\frac{t}{50} + C,$$

where C absorbed the $-\frac{1}{50}$. Exponentiating both sides,

$$100 - \frac{S}{50} = Ce^{-\frac{t}{50}},$$

C replaced e^C. Hence

$$S = 5000 + Ce^{-\frac{t}{50}},$$

C again absorbed a 50. Finally, using the initial conditions of $S(0) = 0$, we get that

$$0 = 5000 + C,$$

and so $C = -5000$. We conclude with the final solution of

$$S(t) = 5000(1 - e^{-\frac{t}{50}}).$$

Now that we have this formula, we can verify what we had guessed earlier. Recall that a function is increasing whenever its derivative is positive. Substituting $S(t)$ into the differential equation, we obtain

$$\begin{aligned} S'(t) &= 100 - \frac{S(t)}{50} \\ &= 100 - \frac{5000(1 - e^{-\frac{t}{50}})}{50} \\ &= 100e^{-\frac{t}{50}} \\ &> 0, \end{aligned}$$

so $S(t)$ is indeed increasing. To check the long-term amount of salt, we compute the limit at $t \to \infty$, that is,

$$\begin{aligned}\lim_{t\to\infty} S(t) &= \lim_{t\to\infty} 5000(1 - e^{-\frac{t}{50}}) \\ &= 5000,\end{aligned}$$

therefore the long-term amount of salt is indeed 5000.

Physical Law (Mixing Formula)**:**
The derivation of the differential equation in the previous example followed from evaluating the change in the amount of salt within a short time. This boils down to setting up an equation like the folloiwng

$$dS = (\text{rate in}) \cdot dt - (\text{rate out}) \cdot dt,$$

where dS is the heuristic notation for the change in S, and dt is the change in time. That is,

$$\frac{dS}{dt} = \text{rate in} - \text{rate out}. \tag{1.1}$$

It can be used to set up a variety of mixing problems as we shall see promptly.

Example 2:
Next, we consider the situation where we eject a rocket straight into space from the surface of the earth and want to determine the speed of the rocket. In the baseball examples we assumed that we are near the surface of the earth and that the gravitational force is constant. If we eject an object into space, this assumption is no longer sensible. To adjust the change in gravitational force, we use Newton's Law of Universal Gravitation which states the force between two point masses is inversely proportional to the square of the distance between them, that is,

$$F_g = \frac{k}{d^2},$$

where $k > 0$ is some constant of proportionality and d is the distance between two objects.

Assume we start the space rocket from the surface of the earth with initial velocity $v(0) = v_0$. We denote by $s(t)$ the distance of the surface of the earth to the space rocket. Then, the distance of the center of the earth to the space rocket is the sum of the radius of the earth, (let's call it) 4000m, and distance from the surface of the earth to the rocket, given by

$$d = s(t) + 4000.$$

Therefore,

$$F_g = \frac{k}{(s(t) + 4000)^2}.$$

For simplicity later on, let's solve for k. Recall that we know F_g when the rocket is on the surface of the earth, that is, at $t = 0$, $s(0) = 0$, and $F_g = -mg$. Hence

$$-mg = \frac{k}{(0 + 4000)^2},$$

(since $s(0) = 0$) and so

$$k = -mg4000^2.$$

Substituting back in, see that

$$F_g = \frac{-mg4000^2}{(s(t) + 4000)^2}.$$

For simplicity we will assume (again) that this is the only force acting on the rocket. Therefore by Newton's Second Law, we have that

$$m\frac{dv}{dt} = F_g = -\frac{mg4000^2}{(s(t) + 4000)^2}.$$

Dividing out the m, we get

$$\frac{dv}{dt} = -\frac{g4000^2}{(s(t) + 4000)^2}.$$

So far this is *not* a differential equation, but v is the speed of the rocket ship and s is the position of this rocket ship, so $v = \frac{ds}{dt}$ and hence

$$\frac{d^2s}{dt^2} = -\frac{g4000^2}{(s(t) + 4000)^2}.$$

Now this *is* a differential equation for the distance of the rocket to the surface of the earth. Note that in contrast to our previous (solved) examples, it involves a second derivative. Solving differential equations with second derivatives are bit trickier and we'll get to a few methods for that later on in the course. In this case, the trick is to note that the gravitational force depends only on the distance to the surface of the earth and not time, t. Hence, we use the chain rule to write

$$\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds}v.$$

Thus, we can rewrite our differential equation as

$$\frac{dv}{ds}v = -\frac{g4000^2}{(s + 4000)^2}.$$

Isolating the derivative terms as usual, we get

$$\frac{dv}{ds} = -\frac{g4000^2}{v(s+4000)^2}$$

This is a differential equation which involves only a first derivative. However, it is not autonomous since the right hand side involves both the dependent variable v and the (newly) independent variable s. Yet, we can push through the same technique that we have used before to solve autonomous equations: Rewriting and integrating both sides of

$$vdv = -\frac{g4000^2}{(s+4000)^2}ds.$$

This first integral gives

$$\frac{v^2}{2} + C$$

and the second

$$-g4000^2 \int (s+4000)^{-2} ds = \frac{g4000^2}{s+4000} + C.$$

Combining the constants of integration, we get

$$v^2 = \frac{2g4000^2}{s+4000} + C.$$

Using $v(0) = v_0$, we get that

$$v_0^2 = \frac{2g4000^2}{0+4000} + C,$$

and hence

$$C = v_0^2 - 8000g.$$

Substituting back in and taking the square root to solve for v, we get that

$$v(s) = \sqrt{\frac{2g4000^2}{s+4000} + v_0^2 - 8000g}.$$

Note that we have chosen the positive square root since we know that the space rocket is rising.

Furthermore, note this solution is a function $v(s)$ instead of $v(t)$. However, since $v(s) = \frac{ds}{dt}(t)$, we have another differential equation of the form

$$\frac{ds}{dt} = \sqrt{\frac{2g4000^2}{s+4000} + v_0^2 - 8000g}$$

which is autonomous. To solve for $v(t)$, we need to actually solve the above differential equation for $s(t)$ and then take the derivative with respect to time. However, as the above turns into quite the difficult integration problem, we shall stop here and just be satisfied with velocity function dependent on the position, since the position of the rocket from the earths surface is a measurable quantity itself.

Example 3:
Let us consider the following variation of the previous mixing problem that will lead to a differential equation which is not separable. As before, the tank contains 100L of water which initially contains no salt, and solution enters and leaves the tank at a rate of 2L/min. However, this time, the inflowing solution contains $2e^{-t}$ grams of salt per liter, t minutes after the system has been started. Using the Mixing Formula Equation (1.1), we have that

$$\begin{aligned}\frac{dS}{dt} &= \text{rate in} - \text{rate out}\\ &= \left(2e^{-t}\frac{\text{g}}{\text{L}}\right)\cdot\left(2\frac{\text{L}}{\text{min}}\right) - \left(\frac{S(t)}{100}\frac{\text{g}}{\text{L}}\right)\cdot\left(2\frac{\text{L}}{\text{min}}\right)\\ &= \left(4e^{-t} - \frac{S(t)}{50}\right)\frac{\text{g}}{\text{min}},\end{aligned}$$

where $S(t)$ denotes the salt in the "well-mixed" mixture. Note that the units of all three terms is g/min. Generally, equal quantities need to have identical units, and one can only add (and subtract) quantities which have identical units. You can use this to check if the differential equation you have set up is reasonable.

2 Population Dynamics and Stability

Differential equations arise quite naturally in the study of population dynamics, which is the study of how the size of a population varies in different situations.

Example 4 (Malthusian Model)**:**
Assume that we have a population of bacteria which multiplies by cell division. It is reasonable to assume that the change of the number of bacteria is proportional to the size of the population. Let us denote the number of individuals at a time t, by $P(t)$. Then the above model can be formalized as the differential equation

$$\frac{dP}{dt} = \alpha P,$$

where $\alpha > 0$ is some constant. If the initial condition is $P(0) = P_0$, then this equation is easily solved (as it's autonomous) to give

$$P(t) = P_0 e^{\alpha t}.$$

2.1 Quantitative Analysis

Sometimes one can actually extract quite a bit of important information from autonomous differential equations without solving them, as we recall from our topic on direction fields. These techniques are useful when dealing with more complicated models which cannot be solved.

Any constant solution to an autonomous differential equation $y' = f(y)$ is called an *equilibrium solution*. The name derives from the fact that if $y = a$ is a constant solution, then if we start at $y(0) = a$, we stay at that level, i.e., the system is in equilibrium. Recall that since equilibrium solutions are constant, they satisfy $\frac{dy}{dt} = 0$. Conversely, we can identify equilibrium solutions by identifying which constant solutions satisfy this differential equation.

For example, in the Malthusian model, suppose that

$$0 = \frac{dP}{dt} = \alpha P.$$

Then we see that the only constant solution satisfying this equation is $P = 0$. This solution corresponds to the case where there are no bacteria initially, and therefore never any bacteria.

Next, one can make quantitative statements about the slope of the solution of the differential equations. Again, using the Malthusian model, since the number of individuals, P, is a non-negative quantity, and $\alpha > 0$, we have that

$$\frac{dP}{dt} = \alpha P \geq 0,$$

and hence all the solutions are non-decreasing. This is not surprising, since the model does not encompass any limiting factors.

In a similar way, we can use the differential equation to determine when the solutions are concave and convex, respectively (sometimes, mathematicians refer to concave by "concave down" and convex by "concave up", you probably learned that in calculus). To do this, we need to analyze the second derivative, which can be computed as follows

$$\frac{d^2P}{dt^2} = \frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{d}{dt}(\alpha P) = \alpha\frac{dP}{dt} = \alpha(\alpha P) = \alpha^2 P \geq 0,$$

where the second and fourth equality used the differential equation. This tells us that all solutions are convex.

One can use this quantitative information to sketch the direction field, even without having actual numbers to compute. Indeed, since our differential equation is autonomous, we know the slope is independent of the t-variable. Moreover, by knowing where the integral curves are increasing, decreasing or constant we can sketch the correct orientation of the slope fields. Then finally, by knowing the convexity of the integral curves, we can conclude the change in steepness of the slope fields. All of this can be done without explicit inputs (compare this with our previous use of direction fields).

The Malthusian model is, in general, a very poor model to explain population growth. The reason is that is poorly predicts the size of the population after a long time, that is,

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} P_0 e^{\alpha t} = \infty.$$

Let's investigate an improved model, namely, the logistic model.

Example 5 (Logistic Equation)**:**
We will improve the Malthusian model by including an additional factor $f(P)$ into the RHS of the differential equation

$$\frac{dP}{dt} = \alpha f(P)P.$$

This factor is supposed to capture the following behavior:

- When the population is small, the population grows nearly proportional to the size of the population, that is, $f(P) \approx 1$, when P is small.
- As the population approaches a certain level, K, (called the carrying capacity) the growth comes to a halt, i.e., if $P \approx K$, then $f(P) \approx 0$.
- If the population exceeds said level K,the population decreases. Formally, if $P > K$, then $f(P) < 0$.

There are several choice for $f(P)$ that satisfy these criteria. The simplest choice is to simply pick a linear function which satisfies $f(0) = 1$ and $f(K) = 0$, such as $f(P) = 1 - \frac{P}{K}$. This choice leads to the logistic equation

$$\frac{dP}{dt} = \alpha P\left(1 - \frac{P}{K}\right).$$

Let's determine the equilibrium solutions of the logistic equation. Assume that P is an equilibrium solution, that is, P is a constant solution. So $\frac{dP}{dt} = 0$, and hence

$$0 = \frac{dP}{dt} = \alpha P\left(1 - \frac{P}{K}\right).$$

As the RHS is a product, it's only zero if one of the factors is zero. The factor $\alpha P = 0$ when $P = 0$ and the factor $(1 - \frac{P}{K}) = 0$ when $P = K$, so we have the two equilibrium solution, $P = 0$ and $P = K$.

Next, let's examine where the solution is increasing or decreasing. The RHS of the differential equation,

$$\frac{dP}{dt} = \alpha P\left(1 - \frac{P}{K}\right),$$

Figure 1: Direction Field of the Logistic Equation with $\alpha = 2$, $K = 1$

consists of two factors which we will examine individually. The following table displays where each of the terms are positive and negative:

	$(0, K)$	(K, ∞)
αP	$+$	$+$
$1 - \frac{P}{K}$	$+$	$-$
$\frac{dP}{dt}$	$+$	$-$

,

we therefore see that the solution is increasing on $(0, K)$ and decreasing on (K, ∞).

We can carry out a similar analysis with the second derivative to examine where the solution is convex and concave. First, let's compute the second derivative by differentiating the differential equation:

$$\begin{aligned}\frac{d^2P}{dt^2} &= \frac{d}{dt}\frac{dP}{dt} \\ &= \frac{d}{dt}\left(\alpha P\left(1 - \frac{P}{K}\right)\right) \\ &= \alpha\frac{dP}{dt}\left(1 - \frac{P}{K}\right) - \frac{\alpha P}{K}\frac{dP}{dt} \\ &= \alpha\left(1 - \frac{2P}{K}\right)\frac{dP}{dt} \\ &= \alpha^2 P\left(1 - \frac{2P}{K}\right)\left(1 - \frac{P}{K}\right).\end{aligned}$$

Note that we have a new zero, namely at $P = \frac{K}{2}$. Again, we consider each term on the on the RHS separately.

	$(0, K/2)$	$(K/2, K)$	(K, ∞)
$\alpha^2 P$	+	+	+
$\left(1 - \frac{2P}{K}\right)$	+	-	-
$\left(1 - \frac{P}{K}\right)$	+	+	-
$\frac{d^2P}{dt}^2$	+	-	+

We conclude that on $(0, K/2) \cup (K, \infty)$, we have that $\frac{d^2P}{dt^2} > 0$ and hence that the solution is convex. On the other hand, we have that $\frac{d^2P}{dt^2} < 0$ on $(K/2, K)$, so on this interval the solution is concave. Recall that convex implies the slopes are increasing in steepness, and concave is decreasing in steepness. We can now sketch the direction field.

Figure 2: Direction Field of the Logistic Equation with $\alpha = 1$, $K = 2$

2.2 Classification of Equilibrium Solutions

Based on the direction field that we have drawn, we can see that for the equilibrium solutions of the logistic equation, there is a difference in how solutions close to the equilibrium solutions behave. On the one hand, for $P = 0$, even if we start very close to $P(0) = 0$, the solutions will finally get very close to the other equilibrium solution $P = K$. On the other hand, if we start very far away from $P(0) = K$, we end up being very close to $P = K$. More precisely, if we start at $P(0) = 0$, we follow the equilibrium solution $P = 0$, and for any other positive initial value $P(0) > 0$, we end up very close to the equilibrium solution $P = K$. We say that an equilibrium solution is **stable** if it attracts nearby solutions, and we say it is **unstable** if it repels nearby solutions. An equilibrium solution which attracts some solutions and repels others is called **semi-stable**.

Formally, an equilibrium solution $y = a_0$ is stable if for all admissible initial values a sufficiently close to a_0, the solutions y_a satisfy

$$\lim_{t\to\infty} y_a(t) = a_0.$$

An equilibrium solution $y = a_0$ is semi-stable if it is not stable, and either for all larger initial values $a > a_0$ close to a_0 or all smaller initial values $a < a_0$ close to a_0, we have

$$\lim_{t\to\infty} y_a(t) = a_0.$$

An equilibrium solution $y = a_0$ is unstable if it is neither stable nor semi-stable. Note that these conditions only apply to admissible initial values, for example, in a population model, we would only consider nonnegative initial values, and the sufficiently close part just tells that we're within two equilibrium solutions.

The direction field, in turn, can be used to classify equilibrium solutions: An equilibrium solution $y = a_0$ is stable if either for all y close to a_0 with $y > a_0$, $\frac{dy}{dt} < 0$, and for all y close to a_0 with $y < a_0, \frac{dy}{dt} > 0$. That means that that arrows near $y = a_0$ are pointing towards a_0.

Use a picture to define this portion.

An equilibrium solution $y = a_0$ is semi-stable if for all y close to a_0 we have that either $\frac{dy}{dt} > 0$ or $\frac{dy}{dt} < 0$. That means that arrows near $y = a_0$ are pointing away from a_0 on one side, and towards a_0 on the other side.

An equilibrium solution $y = a_0$ is unstable if for all y close to a with $y > a_0$, $\frac{dy}{dt} > 0$ and for all y close to a_0 with $y < a_0$, $\frac{dy}{dt} < 0$. That means that arrows near $y = a_0$ are point away from a_0.

Recall that if $\frac{dy}{dt}$ is positive, the arrows in the direction field are pointing upward, and that if $\frac{dy}{dt}$ is negative, the arrows in the direction field are point downward.

This means we can now classify the equilibrium solutions based on the direction field for the Logistic Equation. Since $\frac{dP}{dt} > 0$ on $(0, K)$, solutions starting in that range drift away from zero, and towards K. Therefore, $P = 0$ is an unstable equilibrium solution (as $P < 0$ doesn't make sense). Since $\frac{dP}{dt} < 0$ on (K, ∞), solutions starting in that range drift towards K, too. Therefore, the equilibrium solution $P = K$ is stable. That is why K is called the carrying capacity of the logistic equation. It is the maximal amount of individuals that the environment can sustain. If there are more individuals, their level decreases towards K.

Now that we've done all of this quantitative analysis, let's actually solve the logistic equation, and verify our analysis. The differential equation is separable, so we need to solve

$$\int \frac{1}{P\left(1 - \frac{P}{K}\right)} dP = \int \alpha dt.$$

The RHS is easy and gives

$$\int \alpha dt = \alpha t + C.$$

For the LHS, let's multiply by $\frac{K}{K}$ to give

$$\int \frac{1}{P\left(1 - \frac{P}{K}\right)} dP = \int \frac{K}{P(K - P)} dP.$$

This is a rational function, so we need to compute its partial fraction decomposition. That is, we need to solve

$$\frac{K}{P(K - P)} = \frac{A}{P} + \frac{B}{K - P},$$

for A and B. Cross-multiplying, this gives the equation

$$\begin{aligned} K &= A(K - P) + BP \\ &= (B - A)P + AK. \end{aligned}$$

So $A = 1$ and $B - A = 0$, that is, $B = 1$ as well. Thus,

$$\begin{aligned} \alpha + C &= \int \frac{K}{P(K - P)} dP \\ &= \int \left(\frac{1}{P} + \frac{1}{K - P}\right) dP \\ &= \ln|P| - \ln|K - P| \\ &= \ln\left|\frac{P}{K - P}\right|. \end{aligned}$$

Solving for P in the usual way, we first get that

$$\frac{P}{K-P} = Ce^{\alpha t}.$$

Moving stuff around gives

$$P(1+Ce^{\alpha t}) = KCe^{\alpha t},$$

and hence

$$P = \frac{KCe^{\alpha t}}{1+Ce^{\alpha t}} = \frac{K}{Ce^{-\alpha t}+1}.$$

Finally, assuming an initial population of $P(0) = P_0$, gives us C, as

$$\begin{aligned} P_0 &= P(0) \\ &= \frac{K}{C+1} \end{aligned}$$

yielding

$$C = \frac{K}{P_0} - 1 = \frac{K-P_0}{P_0}.$$

Thus

$$\begin{aligned} P(t) &= \frac{K}{\frac{K-P_0}{P_0}e^{-\alpha t}+1} \\ &= \frac{KP_0}{(K-P_0)e^{-\alpha t}+P_0}. \end{aligned}$$

To classify the equilibrium solutions, we need to determine $\lim_{t\to\infty} P(t)$. We compute

$$\begin{aligned} \lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \frac{KP_0}{(K-P_0)e^{-\alpha t}+P_0} \\ &= \frac{KP_0}{P_0} \\ &= K, \end{aligned}$$

but we need to be careful here, as the expression $\frac{KP_0}{P_0}$ only makes sense if $P_0 \neq 0$, since otherwise we would be dividing by zero. The computation above is therefore not valid if $P_0 = 0$. In that case, we note that $P(t) \equiv 0$ (as 0 clearly solves the logistic equation). In that case, we note that

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} 0 = 0.$$

We conclude that for any positive initial condition $P_0 > 0$, that

$$\lim_{t\to\infty} P(t) = K,$$

and if $P_0 = 0$, that

$$\lim_{t\to\infty} P(t) = 0.$$

That means, that no matter how close to the equilibrium solution $P = 0$ we start, we will never end up approaching that equilibrium solution. Therefore, $P = 0$ is unstable. For all other initial conditions, we approach the equilibrium solution $P = K$, and is hence stable.

This concludes our discussion of the logistic equation. The logistic equation is an improvement over the Malthusian model we introduced earlier in the sense that the population does not explode, that is,

$$\lim_{t\to\infty} P(t) < \infty.$$

However, there are still unsatisfactory features of it. As we have just seen, as long as the initial population is positive, no matter how small, the model will always predict that the population will approach the carrying capacity K. In reality, however, populations which start off with very few individuals are likely to die out. For example, consider a fish population in the Pacific Ocean which consists of three fish. As the ocean is massive, it's *highly* unlikely that any of the fish will meet and be able to reproduce in their short lifetimes and hence would die out (a contradiction to our logistic model).